Set-based control methods for systems affected by time-varying delay.

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THÈSE DE DOCTORAT

DISCIPLINE : AUTOMATIQUE

Ecole Doctorale « Sciences et Technologies de l'Information des Télécommunications et des Systèmes »

Présenté par :

Nikola Stanković

Sujet :

Méthodes ensemblistes pour la commande des sytèmes affectés par retard variable. Application pour la commande des systèmes en réseau

Soutenue le 20/11/2013

devant les membres du jury :

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Set-based control methods for systems affected by time-varying delay

by Nikola Stanković

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In L'Haÿ-les-Roses,

Nikola Stanković

Résumé étendu

Introduction générale

Le problème traité dans cette thèse consiste à commander un processus basé sur un asservissement affecté par des retards. L'approche utilisée repose sur des méthodes ensemblistes. La plus grande partie de cette thèse est consacrée à une conception de commande active pour la compensation des retards qui apparaissent dans des canaux de communication entre le capteur et correcteur. Ce problème est considéré dans une perspective générale du cadre de commande tolérante aux défauts où des retards variés sont vus comme un mode particulier de dégradation du capteur. Le cas de la transmission de signaux mesurés avec retard pour des systèmes avec des capteurs redondants est également examiné. Par conséquent, un cadre unifié est fourni afin de traiter le problème de commande basé sur la transmission des mesures avec retard qui peut également être fournie par des capteurs qui sont affectés par des défauts abrupts.

En général, les approches pour la conception de commande tolérante aux défauts sont regroupées dans les méthodes passives et les méthodes actives. Les méthodes passives considèrent la conception de la commande qui est robuste par rapport aux défauts. De l'autre côté, un système actif réagit à un défaut détecté et reconfigure de la commande de telle sorte que la stabilité et les performances peuvent être vérifiées. Nous pouvons penser de la même manière de la conception de lois de commande pour l'atténuation du retard. À savoir, la conception d'un correcteur robuste pour faire face à des retards est un problème classique dans l'automatique. En outre, la conception d'une commande active tolérante aux retards existe aussi et elle emploie des prédicteurs. Toutefois, en présence d'une perturbation exogène et non mesurée, l'approche basée sur la prévision ne peut pas correspondre aux spécifications de la commande. Ce type de problème est considéré tout au long de cette thèse.

Dans la deuxième partie de la thèse le concept d'invariance positive est exposé. Cependant, l'invariance positive est restée en quelque sorte incomplètement explorée pour les systèmes à retard. Notamment, en ce qui concerne les systèmes linéaires à temps discret affectés par des retards, il existe deux idées principales dans la littérature existante sur la façon d'aborder le problème de l'invariance positive. La première approche repose sur la réécriture d'un tel système dans l'espace d'état augmenté et de le considérer comme un système linéaire régulier. La seconde approche considère l'invariance dans l'espace d'état initial. Cependant, la caractérisation d'un tel ensemble invariant est encore une question ouverte, même pour le cas linéaire. Par conséquent, l'objectif principal de cette thèse est d'introduire une notion d'invariance générale positive pour les systèmes linéaires à temps discret affectés par des retards. En outre, certains nouveaux éclairages sur l'existence et la construction pour les ensembles invariants positifs robustes sont détaillés. En outre, les nouveaux concepts d'invariance alternative sont également décrits.

Objectif

Les objectifs de cette thèse sont les suivants:

- Caractériser les ensembles D-invariants pour un système linéaire avec une perturbation additive et fournir une méthode numérique pour leur construction.
- Contribuer à la création des conditions nécessaires et suffisantes pour l'existance de la $D-$invariance.
- Fournir un mécanisme de détection de retard basé sur la conception de gouverneur de référence. En outre, afin de développer une méthode de commande active pour la compensation des retards introduits par réseau et qui est robuste par rapport aux perturbations exogènes et bornées.
- Établir un système avec plusieurs capteurs de commande tolérante aux défauts et des retards en utilisant des caractéristiques des ensembles invariants.

Structure de la thèse

Chapitre 2

Une commande comprenant plusieurs modules interconnectés est souvent réalisée par un réseau commun. Cette façon d'interconnexion entre des capteurs, des actionneurs et des correcteurs apporte de nombreux avantages tels que la réduction du câblage du système, la flexibilité et la baisse du coût. Toutefois, le transfert d'informations en paquets, ce qui est une particularité des systèmes commandés par le réseau, présente quelques défis qui ne sont généralement pas pris en compte dans la théorie de la commande classique. Par exemple, en raison d'une contrainte de taille de paquets, des informations limitées peuvent être transférés par un seul paquet. Par conséquent, une grande quantité de données doit être brisée en plusieurs paquets afin d'être transmise. Ces limitations de bande passante représentent une différence majeure par rapport aux systèmes de données échantillonnées conventionnelles où l'on suppose que les données sont transmises en même temps. Une autre question importante liée à la commande en réseau sont les abandons de paquets (ou le rejet de données). A savoir, en raison de liens non fiables, il peut arriver que certains paquets sont perdus lors de la transmission. De plus, pour les applications de commande en temps réel, il est souvent avantageux d'utiliser uniquement des informations qui sont mises à jour au cas où elles sont disponibles. La communication sur un réseau partagé peut être source de retards. En particulier, la principale source de retards dans les systèmes commandés en réseau est le temps nécessaire pour accéder au réseau par un noeud pour transmettre des données. Les retards peuvent également être générés par un temps qui est nécessaire pour le calcul de la commande ou de la transmission de données à travers le réseau. Tous ces effets introduits par le réseau doivent être pris en considération avec précaution car ils peuvent dégrader considérablement les performances de la dynamique en boucle fermée. Nous considérons la modélisation d'un processus en temps continu qui est commandé par un correcteur numérique en réseau partagé.

L'architecture souvent utilisée pour l'analyse de commande en réseau est réprésentée par un système avec la boucle de rétroaction unique (Fig. 1). Afin de transmettre un signal de temps continu sur le réseau, le signal doit être échantillonné et codé dans un format numérique. Des données obtenues sont ensuite transmises via le réseau au dispositif de commande numérique, où l'action de commande est générée. Le signal de commande

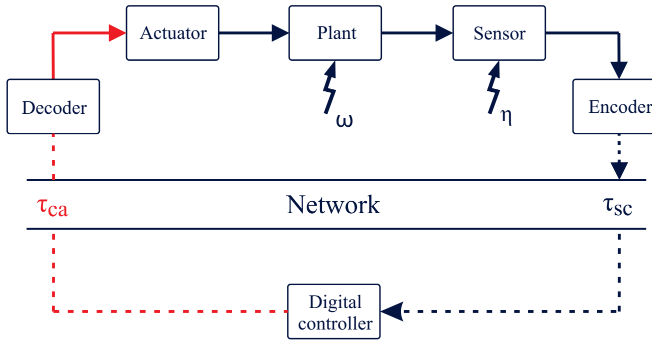


FIGURE 1: Commande en réseau

est décodé au site de l'actionneur et transformé en un signal continu par un bloqueur d'ordre zéro.

Prenons le cas où la communication entre des sous-systèmes est affectée par des retards. A savoir, à $t = t_k$, $k \in \mathbb{Z}_+$, chaque sous-système (capteur et correcteur) nécessite l'autorisation d'accéder au réseau pour transmettre des données. En fonction de la disponibilité du réseau, nous pouvons avoir la situation où un noeud transmet des données, tandis que les autres doivent attendre jusqu'à ce que le réseau est inactif. Sans perte de généralité, tous les retards introduits par le réseau sont considérés comme des retards capteur-correcteur ou correcteur-actionneur.

Supposons que le capteur est déclenché par une horloge, c'est-à-dire, la sortie est mesurée périodiquement à chaque impulsion de l'horloge. Supposons aussi que le correcteur et l'actionneur sont déclenchés par un événement, c'est-à-dire, tous les signaux sont mis en oeuvre dès qu'ils sont reçus. Pour une variation de retard aléatoire, le nombre possible de changements de commande active pendant une période d'échantillonnage est une variable qui dépend du retard (comme paramètre).

Quand le retard est variable dans le temps, l'analyse de la stabilité d'une telle dynamique n'est pas simple et elle est basée sur la recherche d'une fonction de Lyapunov commune pour toutes les représentations du système sur un ensemble infini de paramètres. Afin de déterminer pratiquement une fonction de Lyapunov commune pour un tel système, on peut utiliser une grille finie de la plage de retards, ce qui conduit à un ensemble fini d'inégalités matricielles linéaires. Il est également important de souligner ici que l'existence d'une fonction de Lyapunov ne représentant qu'une condition suffisante de stabilité, et donc, elle peut représenter un résultat limité.

Une approche qui fournit une approximation finie est développée en utilisant un tampon au niveau du correcteur qui est situé sur le site de l'actionneur. Le tampon est lu périodiquement à une fréquence plus élevée que la fréquence d'échantillonnage du capteur. Un modèle NCS obtenu de cette façon a un nombre fini de configurations différentes, c'est-à-dire, il est représenté par une dynamique de commutation.

En utilisant un tampon dans la boucle de régulation, la réaction du correcteur et de l'actionneur est retardée par rapport au cas où ces noeuds sont uniquement gérés par des événements. En d'autres termes, toutes les mises à jour ne sont prises en considération que si le tampon est lu. Toutefois, en choisissant un échantillonage du tampon sufisamment petit on obtient un correctuer et un actionneur qui réagissent plus vite. D'un autre côté, une période d'échantillonnage relativement faible va fournir un modèle du système en boucle fermée qui est plus simple.

En tenant compte des différences entre τ_{sc} et τ_{ca}, nous pouvons discerner deux directions principales afin de s'attaquer à des retards introduits par le réseau. La première direction représente l'approche de commande robuste par rapport aux retards. L'objectif principal selon cette approche serait de concevoir une action de la commande qui garantit la stabilité du système en boucle fermée pour tout variation du retard. Cette approche utilise typiquement la méthode de stabilité de Lyapunov et elle est largement traitée dans la littérature. La deuxième direction, qui est moins examinée est de considérer une stratégie active avec la détection de retard et de reconfiguration de commande afin de réaliser la compensation du retard. Une telle approche peut être considérée comme la commande tolérante aux défauts avec la détection des défauts et avec un mécanisme pour reconfigurer de la commande afin que la stabilité et les performances peuvent être vérifiées en présence des retards. On peut remarquer que de telles mesures de la commande ne s'appliqueraient qu'aux retards capteur-correcteur. Une architecture NCS spécifique qui correspond à ce cas est représentée dans la Fig. 2, où le correcteur colocalise avec l'actionneur.

Il a été démontré que la performance de la NCS dépend en grande partie du protocole de réseau sous-jacent. Dans cette section, nous considérons la modélisation pour le NCS, avec des dynamiques linéaires, en ce qui concerne les retards introduits par le réseau et les abandons de paquets. Les modèles obtenus sont basés sur l'hypothèse que les mesures obsolètes sont rejetées. Même si la mise en oeuvre de cette approche exigerait des protocoles réseau non standard, une telle approche serait plus appropriée pour les

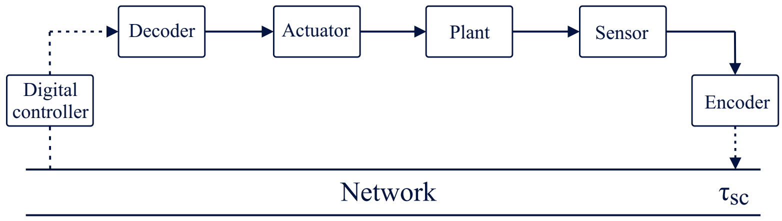
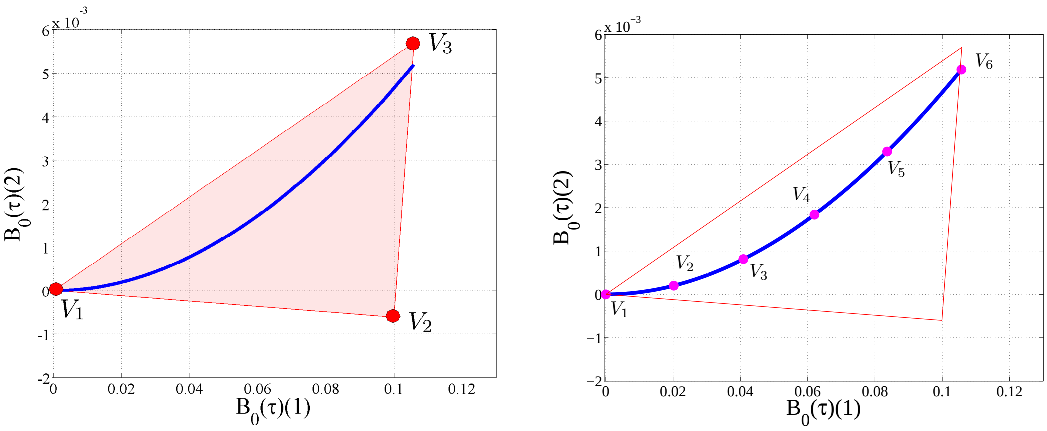


FIGURE 2: Le correcteur colocalise avec l'actionneur

applications de la commande en temps réel. Ainsi, une charge supplémentaire du réseau est évitée.

Nous avons considéré un paramètre de retard qui est incertain et borné sur un intervalle. Cette incertitude est incorporée dans un modèle qui, par conséquent, a un nombre infini de modes possibles. Afin d'obtenir un modèle mathématique adapté à l'analyse de la stabilité, deux approches ont été proposées. La première approche exploite la méthode de surapproximation. A savoir, le paramètre de retard dépendant est délimité par une région convexe (voir Fig. 3a). Par conséquent, la stabilité à l'égard de toutes les variations du retard possibles est garantie par la stabilité des générateurs de cet ensemble convexe. La qualité de l'approximation et le coût numérique sont des exigences contradictoires et elles dépendent du nombre de générateurs. Le principal avantage de cette approche est qu'elle prend en compte tous les retards possibles de l'intervalle à un coût numérique relativement faible. De l'autre côté, cette approche est plus rigoureuse en ce qui concerne la qualité de l'approximation, car elle prend en considération la région large d'incertitude. La seconde approche, basée sur la stratégie de l'sous-échantillonnage, ne nécessite aucune surapproximation puisque les modèles obtenus ont déjà un nombre fini de différents modes de commutation. Cependant, le nombre de ces modes peut être plus grand pour une qualité d'approximation acceptable comme c'est le cas pour la méthode de surapproximation. Le principal avantage de cette approche est qu'elle correspond à la configuration du système, car le tampon introduit permet seulement certains retards de l'intervalle de retard, en particulier, ceux qui sont sur la grille déterminée par le nombre d'sous-échantillons (Fig. 3b).



(A) L'intégration convexe (B) Sous-échantillonnage

FIGURE 3: Rapprochement du paramètre de retard incertain

Chapitre 3

L'invariance positive est un concept largement utilisé dans la commande et elle fournit un moyen efficace pour aborder l'analyse de la commande avec des contraintes. Par exemple, des contraintes physiques imposées sur un système dynamique peuvent être satisfaites en assurant l'existence d'un ensemble positivement invariant à l'intérieur de la région admissible. De cette façon, on définit directement un ensemble de conditions initiales pour lesquelles la dynamique répond à des contraintes pour tous les instants à venir. Cette propriété remarquable d'ensembles positivement invariants trouve des applications dans de nombreux autres domaines de commande comme la commande prédictive, la commande tolérante aux défauts, la synthèse de générateur de référence, etc.

En ce qui concerne les systèmes en temps discret avec des retards, il y a deux idées principales dans la littérature existante sur la façon d'aborder le problème de l'invariance positive. La première approche repose sur la réécriture d'un système dans l'espace d'état augmenté et de le considérer comme un système linéaire régulier. D'autre part, la seconde approche appelée également *D-invariance* considère l'invariance directement dans l'espace d'état initial. Cependant, la caractérisation des ensembles D-invariants est encore un problème ouvert, même pour les systèmes linéaires à temps discret. Nous considérons le problème de conception de la commande prédictive sur un modèle du système en réseau $x[t_{k+1}] = A[t_k] + \sum_{j=0}^{d_m} B_{d_m-j} u[t_{k+j-d_m}]$, $x \in \mathcal{X}_x$, $u \in \mathcal{X}_u$ avec des retards constants et avec des contraintes sur l'état et l'entrée. La séquence de commande sur un horizon fini est obtenue en résolvant le problème d'optimisation suivant:

$$\min_{u} \sum_{s=k}^{k+N-1} l\left(x[t_s], u[t_k], \ldots, u[t_{s-d_m}]\right) + T\left(x[t_{k+N}], u[t_{k+N}] \ldots, u[t_{k+N-d_m}]\right), \quad N > d_m \tag{1}$$

sous les contraintes

$$\begin{gathered} x[t_{s+1}] = Ax[t_s] + \sum_{j=0}^{d_m} B_{d_m-j} u[t_{s+j-d_m}], \quad \forall s \in \mathbb{Z}_{[k,k+N-1]} \\ \begin{bmatrix} x[t_s]^T & \ldots & u[t_{s-d_m}]^T \end{bmatrix}^T \in \mathcal{X}_x \times \ldots \times \mathcal{X}_u, \quad u[t_s] \in \mathcal{X}_u \\ \begin{bmatrix} x[t_{k+N}]^T & \ldots & u[t_{k+N-d_m}]^T \end{bmatrix}^T \in \mathcal{X}, \end{gathered} \tag{2}$$

où $l(\cdot) > 0$, $T(\cdot) > 0$ sont les coûts dans la fonction objectif, tandis que $\mathcal{X}$ est un ensemble de terminaux positivement invariant. Cet ensemble peut être défini dans un espace d'état augmenté : $\mathcal{X} \subseteq (\mathbb{R}^n)^{d_m+1}$ où $\mathcal{X} = \mathcal{Y} \times \ldots \mathcal{Y} \subseteq (\mathbb{R}^n)^{d_m+1}$ et $\mathcal{Y} \subseteq \mathbb{R}^n$ est un ensemble D-invariant. Cette formulation remplit l'objectif initial: faire respecter le confinement des trajectoires dans une structure invariante afin d'assurer la stabilité de la loi de commande.

Il existe deux objectifs principaux de ce chapitre. Le premier consiste à fournir un appui en termes d'ensembles invariants pour un scénario de défaut et des algorithmes de détection du retard qui sont considérés dans le Chapitre 5 et le Chapitre 6. Relatif à ce sujet, les résultats nécessaires sont déjà bien connus dans la littérature et ils comprennent l'invariance positive pour les systèmes linéaires avec une perturbation additive. Le deuxième objectif est d'introduire une notion d'invariance générale positive pour les modèles NCS présentées dans le chapitre précédent. A cet effet, nous donnons un aperçu des différents concepts d'invariance pour les systèmes en temps discret avec des retards. En outre, nous détaillons quelques nouvelles idées sur l'existence et sur la construction des ensemble D-invariants. En outre, certaines solutions d'invariance alternative sont discutées aussi.

Pour la conception de la commande tolérante aux défauts, nous utilisons principalement l'ensemble invariant positif robuste minimal. Le calcul d'une représentation exacte de l'ensemble invariant positive robuste minimal n'est pas possible dans le cas général. Il faut alors recourir à des approximations et des algorithmes différents pour la construction d'approximations. Quelques résultats récents offrent une approche itérative pour calculer une approximation avec une précision arbitraire au prix d' une complexité accrue.

Indépendamment de leur application générale dans la conception de commande avec des contraintes, des ensembles positifs invariants robustes minimaux sont également utilisés pour la caractérisation de l'effet du bruit de mesure sur la dynamique en boucle fermée. Bien sûr, au lieu de l'ensemble robuste positif invariante minimal, on peut aussi utiliser des limites ultimes. Cependant, l'ensemble positif invariant robuste minimal détermine la règion invariante la plus petite possible dans l'espace de l'état que l'on peut avoir.

Dans la deuxième partie de ce chapitre, l'analyse de la stabilité de l'équation récurrente linéaire en temps discret avec le retard est effectuée de la manière la plus simple par l'introduction d'un nouveau vecteur d'état qui est composé du vecteur d'état initial et de tous les signaux d'entrée de la fenêtre du retard. La représentation d'état augmentée ainsi obtenue est linéaire et sa stabilité peut être vérifié soit en utilisant la méthode directe de Lyapunov, soit en vérifiant les valeurs propres de la matrice du système en boucle fermée. La même méthode peut être appliquée pour la conception de la commande.

Nous avons montré que l'invariance positive sur l'espace d'état initial est une propriété plus forte que la stabilité de la représentation d'état augmentées. Par conséquent, nous nous référons à l'invariance positive dans l'espace d'état initial comme $D-$*invariance*. Bien que le concept d'invariance positive pour les représentations d'état augmentées soit déjà bien établi dans la littérature, nous mettons l'accent sur la $D-$invariance dans ce chapitre.

Malgré les progrès vers la caractérisation de D-invariants, des résultats assez basiques sont toujours manquants, par exemple l'existence de D-invariants n'est pas complètement caractérisée. Dans ce travail, nous ne fournissons pas de réponse complète à ce problème non plus. Mais dans reste de ce chapitre, nous abordons le problème de l'existence et de la construction de l'ensemble robuste $D-$invariant minimal pour l'équation récurrente en temps discret avec retard affecté par des perturbations additives. Nous fournissons également quelques nouveaux concepts d'invariance comme l'invariance cyclique, la famille d'ensembles invariants et les ensembles invariants paramétrés.

Ce chapitre a été consacré à l'invariance positive pour les systèmes en temps discret, où l'on a examiné cette notion en ce qui concerne l'espace d'état initial. S'il exist un ensemble positivement invariant défini pour l'espace d'état initial, il est préférable en raison de sa représentation simple. En guise de contribution à l'égard de l'état de l'art, nous avons fourni les résultats sur la caractérisation de l'ensemble positif invariant robuste minimal. Dans la dernière section de ce chapitre un nouvel aperçu est disponible sur des solutions alternatives pour l'invariance par rapport à l'équation récurrente en temps discret avec le retard. En outre, il a été montré que la factorisation des ensembles convexes est l'opération permettant la description d'ensembles invariants dans des espace d'état de dimensions différentes. Cette flexibilité ouvre de nouvelles perspectives pour une meilleure gestion de la complexité des contraintes décrivant les ensembles invariants. Par la suite, les ensembles invariants de faible complexité ont des répercussions sur la complexité de la syntèse de la commande comme par exemple dans le cadre de la commande prédictive.

Chapitre 4

L'objectif de ce chapitre est de caractériser l'existence d'ensembles invariants positifs pour une équation récurrente linéaire en temps discret à retard variable (dDDE). L'angle considéré dans ce chapitre est complètement différent par rapport aux approches existant dans la littérature. Afin de réduire le conservatisme des méthodes dans le domaine temporel, dans cette étude nous utilisons le une approche dans le domaine fréquentiel. En particulier, une notion forte de stabilité des équations récurrentes linéaires en temps discret à retard variable, notée comme stabilité asymptotique robuste, et sa relation avec la D-invariance est examinée. Cette notion est nommée “forte”, car elle définit la stabilité à l'égard de toutes les réalisations du retard. Nous nous tournons vers une classe plus générale, celle qui est spécifiée dans le domaine en temps continu.

Les équations récurrentes linéaires en temps continu au retard variable (cDDE) sont largement traitées dans la littérature, surtout dans le contexte de l'équation différentielle fonctionnelle où elles jouent un rôle important dans l'analyse de la stabilité. Une particularité de la cDDE est sa “sensibilité” pour une perturbation de retard arbitrairement petite. Dans ce chapitre, le concept de stabilité forte est notée comme la stabilité indépendante du retard. L'importance de la cDDE est une nouvelle idée suggérant que cette classe de systèmes permet l'analyse de la stabilité asymptotique robuste et l'existence de D-invariants.

Dans ce chapitre nous considérons la stabilité des équations récurrentes linéaires à retard variable en temps discret. Le problème de la stabilisation n'est pas traitée ici, donc nous allons analyser le modèle suivant:

$$x[t_k] = \sum_{i=1}^{m} A_i x[t_{k-d_i}], \tag{3}$$

avec $x[t_{k-d_i}] \in \mathbb{R}^n$, $A_i \in \mathbb{R}^{n\times n}$ et $d_i \in \mathbb{Z}_{[1,\infty]}$ telle que $0 < d_i < d_{i+1},\ \forall i \in \mathbb{Z}_{[1,d_m]}$.

Tous les retards constituent un vecteur $\vec{d} = [d_1 \dots d_m]^T \in (\mathbb{Z}^+)^m$ dans l'espace du paramètre $\vec{d}$. Pour chaque $\vec{d}$ nous définissons un rayon discret

$$\mathcal{T}_d(\vec{d}) = \{\alpha\vec{d}\ :\ \alpha \in \mathbb{Z}^+\}. \tag{4}$$

Pour estimer la stabilité asymptotique de l'équation (4.1) on peut augmenter l'espace d'état en réécrivant tous les états en retard comme un vecteur colonne. Sans perte de généralité, supposons que $\vec{d} = \begin{bmatrix}1 & 2 & \dots & m\end{bmatrix}$. Ensuite, la représentation d'état augmentée s'écrit:

$$X[t_k] = \mathbb{A}(\vec{d})X[t_{k-1}] = \begin{bmatrix} A_1 & \dots & A_{m-1} & A_m \\ I_n & \dots & 0_{n\times n} & 0_{n\times n} \\ \vdots & \ddots & \vdots & \vdots \\ 0_{n\times n} & \dots & I_n & 0_{n\times n} \end{bmatrix} X[t_{k-1}], \tag{5}$$

avec $X[t_{k-1}] = \begin{bmatrix} x[t_{k-1}]^T & x[t_{k-2}]^T & \dots & x[t_{k-m}]^T \end{bmatrix}^T$.

Notons par z^* toutes les valeurs propres de $\mathbb{A}(\vec{d})$. Supposons d'abord que les retards dans (4.1) varient de telle manière que le vecteur de retard reste toujours sur le même rayon, c'est à dire, $\vec{d} \in \mathcal{T}_d(\vec{d})$. La stabilité de (4.1) par rapport à cette variation de retard est caractérisée par le spectre suivant :

$$\sigma(A(\alpha\vec{d})) = \left\{ z^*_\alpha\ :\ \det\left(I_{nm} - \sum_{i=1}^{m} A_i z^*_\alpha = 0 \right),\ \alpha \in \mathbb{Z}^+ \right\}. \tag{6}$$

On peut remarquer que $z^* = z^*_\alpha$. Par conséquent, si la solution nulle de l'équation (4.1) est asymptotiquement stable pour un vecteur $\vec{d}$, elle restera asymptotiquement stable $\forall \vec{d} \in \mathcal{T}_d(\vec{d})$. On note un rayon avec cette propriété comme stable.

Du point de vue d'invariance positive, nous nous sommes plus intéressés à la stabilité qui concerne la variation du retard (variation sur tous les rayons).

Lorsque la stabilité asymptotique robuste est examinée, on peut s'apercevoir que le domaine temporel discret est en quelque sorte “incomplet”. Notamment, on peut remarquer que pour une petite perturbation d'un rayon stable, la dDDE (4.1) peut devenir instable.

Example 0.1. Compte tenu de la dDDE stable :

$$x[t_k] = 3/4x[t_{k-d_1}] - 1/2x[t_{k-d_2}]. \tag{7}$$

Pour $d_1 = 1$ et $d_2 = 2$ toutes les valeurs propres sont à l'intérieur du cercle unité, c'est à dire, $|z_{max}| = 0.7071$, avec

$$|z_{max}| = \max_i\{|z_i| \ : \ \det(1 - \frac{3}{4}z^{-d_1} + \frac{1}{2}z^{-d_2})\}.$$

Pour $\alpha = 10$, c'est à dire, pour $d_1 = 10$ et $d_2 = 20$, le système est stable (retards du même rayon), mais avec la plus petite marge de stabilité. En effet, $|z_{max}| = 0.9659$. La même chose vaut également pour $d_1 = 11$ et $d_2 = 22$, avec $|z_{max}| = 0.9690$. Toutefois, une petite perturbation de la direction de ce rayon peut causer de l'instabilité. Par exemple, pour $d_1 = 10$ et $d_2 = 21$ nous obtenons $|z_{max}| = 1.0159$.

L'exemple précédent affiche une sensibilité de la stabilité pour la dDDE par rapport à une petite variation de retard. Il est clair qu'on ne peut pas gérer correctement ce phénomène à l'aide de la représentation en temps discret. Pour cette raison, nous considérons la classe générale des équations récurrentes linéaires en temps continu à retard variable (cDDEs).

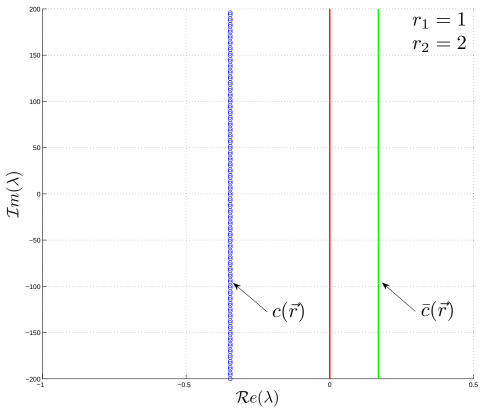
Considérons la cDDE:

$$x(t) = \sum_{i=1}^{m} A_i x(t - r_i), \tag{8}$$

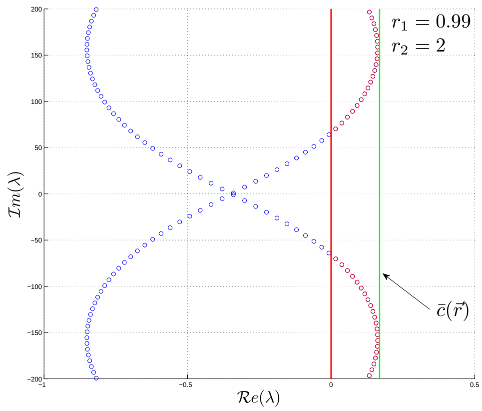
avec $t \in \mathbb{R}_+$, $A_i \in \mathbb{R}^{n\times n}$, $r_i \in \mathbb{R}^+$, $i \in \mathbb{Z}_{[1,m]}$. Il est clair que (4.1) est obtenue comme un cas particulier de (4.14) en limitant le temps continu variable t pour les séquences en temps discret t_k. Pour chaque condition initiale $\varphi \in \mathcal{C}_D(\mathbb{R}_{[-r_m,0]}, \mathbb{R}^n)$, une solution de (4.14) sur $\mathbb{R}_{[-r_m,0]}$ est définie de façon unique.

Pour organiser des retards dans un vecteur, nous définissons $\vec{r} = [r_1 \dots r_m]^T \in (\mathbb{R}^+)^m$, avec $r_i < r_{i+1}$. Pour chaque $\vec{r}$ il est possible de définir un *rayon* :

$$\mathcal{T}_c(\vec{r}) = \{\beta\vec{r} \ : \ \beta \in \mathbb{R}^+\}. \tag{9}$$



(A) $r_1 = 1, \ r_2 = 2$



(B) $r_1 = 0.99, \ r_2 = 2$

FIGURE 4: Les valeur propres sans et avec la perturbation du retard

De la même manière que pour l'equation en temps discret, si le cDDE (4.14) est asymptotiquement stable pour un vecteur $\vec{r} \in \mathcal{T}_c(\vec{r})$, il reste aussi asymptotiquement stable $\forall \vec{r} \in \mathcal{T}_c(\vec{r})$. Un tel rayon est noté comme stable. Cependant, la stabilité d'un rayon est une "propriété sensible" pour les cDDEs.

Example 0.2. Compte tenu de la dDDE stable :

$$x(t) = 3/4x(t - r_1) - 1/2x(t - r_2). \tag{10}$$

Pour $r_1 = 1$ et $r_2 = 2$, toutes les valeurs propres sont stables (voir Fig. 5.9a). Notez que les pôles pour le cas temps discret et le cas continu sont liés par la transformation $e^{\lambda} = z$. Ainsi, pour chaque valeur propre dans le $z-$domaine complexe il existe un nombre infini de la valeur propre dans le $\lambda-$domaine complexe avec la même partie réelle et des arguments avec une périodicité de 2π.
De l'autre côté, pour $r_1 = 0.99$ et $r_2 = 2$ le spectre de (4.20) est représenté dans la Fig. 5.9b.

Par la suite, la corrélation entre la stabilité asymptotique robuste et l'existence des ensembles D-invariants est adressée. Le lien entre le temps discret et la dynamique en temps continu devient naturel si la notion de D-invariance est définie par rapport à la dynamique en temps continu. Nous avons également montré que, à côté de leur application pratique, des ensembles D-invariants peuvent également apporter un nouvel éclairage sur la corrélation entre la robustesse et la stabilité asymptotique qui est indépendante du retard.

Chapitre 5

Connaissant la valeur d'un retard introduit dans la communication entre le capteur et le correcteur, un modèle suffisamment précis de ce processus nous permet de concevoir une action de commande qui est capable de compenser ce retard. Une approche classique utilise une commande basée sur un modèle, qui effectue l'estimation d'un vecteur d'état basée sur des mesures passés et des entrées de commandes précédentes. Toutefois, si le processus n'est pas déterministe (il est soumis à des perturbations additives ou des incertitudes paramétriques), cette approche peut fournir des performances médiocres, ou même conduire à un comportement instable. D'autre part, un retard qui pourrait apparaître dans la communication entre les correcteurs et les actionneurs a une nature complètement différente. Sauf au cas où il peut être estimé à l'avance et pris en compte dans la conception de la commande, un tel retard ne peut pas être compensé par des moyens de commande en réseau. Dans cette situation, il faut compter sur la robustesse de la commande (par rapport au paramètre de retard).

Ce chapitre examine une commande des systèmes en réseau avec un seul canal de communication capteur- correcteur. Les retards aléatoires et variables, ce qui peut se produire lors de la transmission de données à travers ce canal, sont considérés comme des défauts. En outre, nous supposons que le processus est affecté par une entrée de perturbation bornée.
Afin de fournir des informations sur les retards plus faibles que la période d'échantillonnage, nous concevons un mécanisme de détection basé sur des ensembles invariants. Un avantage du mécanisme proposé est sa mise en oeuvre simple car il utilise des tests ensemblistes pour discerner une information "saines" d'une information "retardées". Afin d'éviter intersection entre des régions "saine" et "retardé" un générateur de référence est conçu en utilisant le cadre de la commande prédictive. Une fois que les informations sont fournies par le mécanisme de détection, le correcteur calcule une mesure de commande fondée sur la prévision. Toutefois, si le système est affecté par la perturbation, le signal de commande ainsi obtenu peut présenter une erreur de suivi qui s'accorde à chaque nouvelle variation de retard. Pour cette raison, nous avons conçu une commande basée sur un modèle avec un bloc de compensation qui est capable de corriger l'erreur induite

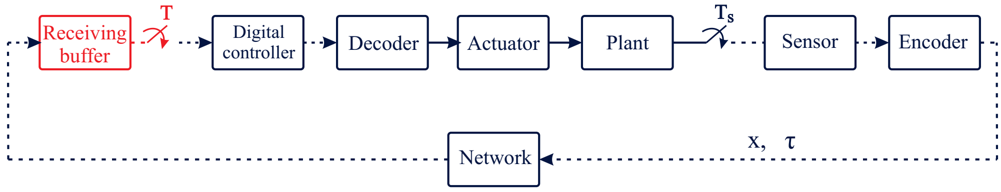


FIGURE 5: Commande en réseau avec le correcteur qui est colocalisé avec l'actionneur

de suivi. Une condition suffisante qui garantit l'existence d'un signal de compensation est également prévue.

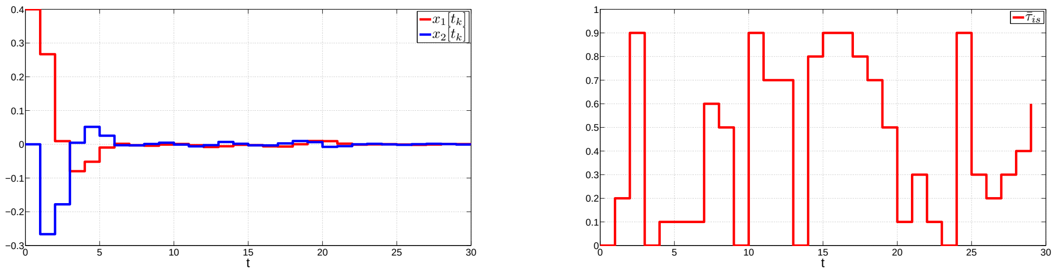
Le modèle du système de commande en réseau qui est considéré dans cette section a un correcteur colocalisé avec l'actionneur (voir Fig. 5).

Idéalement, à chaque période d'échantillonnage l'action de la commande doit être mise à jour en fonction des dernières mesures acquises par le capteur. Cependant, parce qu'il est partagée par plusieurs noeuds, le réseau de communication peut ne pas être disponible au moment où il est requis par le capteur. Quand cela arrive, les paquets de données sont mis en attente jusqu'à ce que le protocole de réseau accorde la permission pour leur transmission. Pour la plupart les protocoles des retards induits sont aléatoires et variables. Les données transmises sont stockées dans un tampon de réception qui est lu par le dispositif de commande à une fréquence plus élevée que la fréquence d'échantillonnage du capteur.

La même dynamique peut avoir des réponses différentes pour les différentes variations de retard. A titre d'exemple nous pouvons remarquer quelques résultats d'une simulation représentée dans la Fig. 6 et dans la Fig. 7.

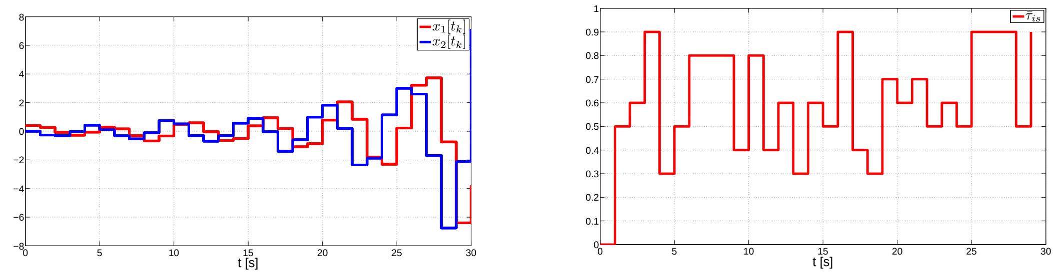
Un défaut est défini comme une déviation de la structure du système ou des paramètres du système à partir de la spécification nominale. Suivant la même idée, on peut considérer un retard introduit par le réseau comme une déviation dans les canaux de communication à l'égard de la transmission de données. Cette dernière voie représente la principale approche que nous examinons dans ce chapitre.

Afin de calculer le signal de commande, nous considérons l'asservissement d'état et l'asservissement d'état estimé. Les deux cas seront souvent considérés simultanément en utilisant une notation commune. Cependant, certaines notions ne sont pas totalement compatibles et pour ces cas, le retour d'état et le mecanisme d'estimation doivent être examinés séparément.



(A) Réponse en boucle fermée (B) La variation de retard

FIGURE 6: Réponse en boucle fermée stable



(A) Réponse en boucle fermée (B) La variation de retard

FIGURE 7: Réponse en boucle fermée instable

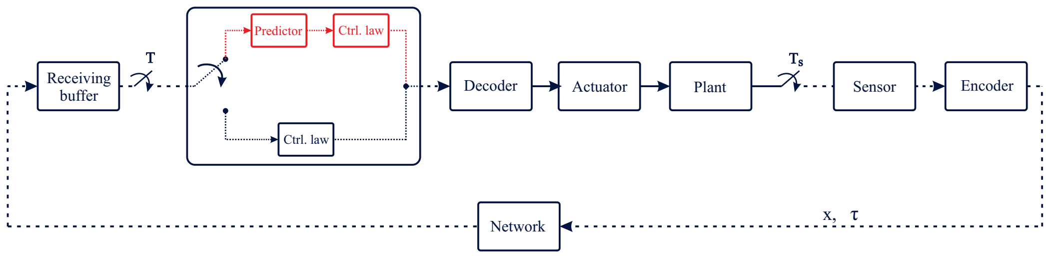
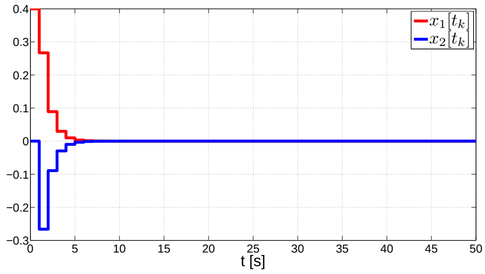
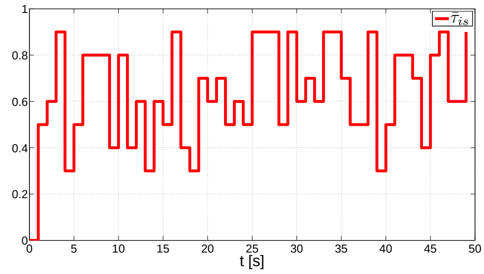


FIGURE 8: Correcteur basé sur un modèle du système

Pour faire face aux retards introduits dans la communication entre le capteur et le correcteur, une approche classique utilise une commande basée sur la prédiction. Un tel correcteur utilise le modèle mathématique du processus afin d'estimer le vecteur d'état courant à partir des mesures obsolètes disponibles. Dans cette section, nous considérons un correcteur numérique de commutation avec deux boucles de régulation

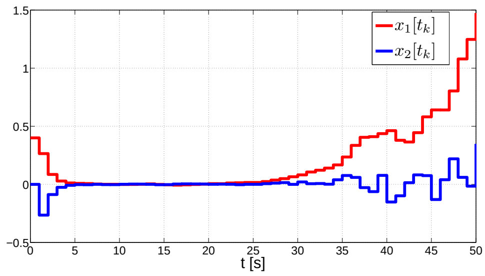


(A) Réponse en boucle fermée

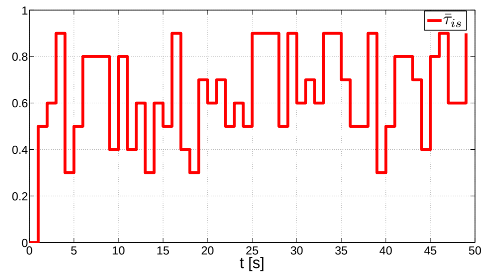


(B) La variation de retard

FIGURE 9: Réponse en boucle fermée quand $\omega = 0$



(A) Réponse en boucle fermée



(B) La variation de retard

FIGURE 10: Réponse en boucle fermée quand $\|\omega\|_\infty \leq 0.004$

(voir Fig. 8.). Si le correcteur est fourni avec les mesures obsolètes, la commande est générée en fonction des informations fournies par le prédicteur (boucle de commande en haut dans la Fig. 8.). D'un autre côté, lorsque les mesures mises à jour sont disponibles, le dispositif de commande est commuté sur la deuxième boucle de régulation.

Une éventuelle limitation de cette approche pourrait être originé par les deux hypothèses suivantes. Le modèle suffisamment précis du processus est connu; si cette hypothèse ne tient pas, le fonctionnement global de la stratégie de commande peut être affecté. Ici, nous examinons principalement les questions liées aux incertitudes du processus. Par conséquent, dans cette section, nous supposons que tous les retards entre le capteur et le correcteur sont connus. En conséquence, la même dynamique en boucle fermée, contrôlée par un correcteur basé sur un modèle, peut avoir des réponses différentes en présence de perturbations (voir Fig. 9 et Fig. 10)

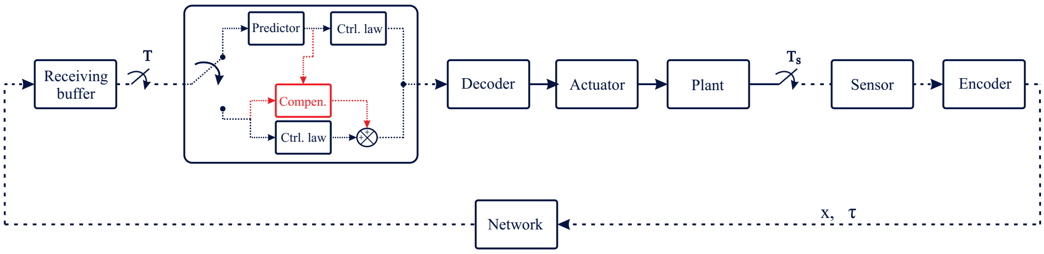


FIGURE 11: Correcteur basé sur modèle avec compensation active du retard

La conception du compensateur de retard exploite l'idée qu'une occurrence du retard peut être considérée comme une faute à l'égard de la dynamique nominale. Afin de réduire l'écart de suivi dans ce cas, nous concevons un mécanisme FDI et la reconfiguration de la commande. Le schéma bloc de cette stratégie est présenté dans la Fig. 11.

Chapitre 6

L'utilisation de capteurs redondants dans la boucle de commande est inévitable dans les applications critiques pour la sécurité. Par exemple, afin d'améliorer la sécurité dans l'aéronautique, les avions sont équipés avec plusieurs systèmes pour mesurer la vitesse, le nombre de Mach et l'altitude. Alors qu'initialement des systèmes avec des composantes redondantes ont été limités par la hausse des coûts, de nos jours, avec une baisse continue des coûts de production et d'exploitation la redondance des capteurs est devenue accesible dans de nombreux autres domaines. A titre d'exemple, on peut citer le régulateur automatique de la vitesse dans les voitures, qui est basé sur des mesures multi- capteurs. L'objectif principal de la redondance des capteurs dans ces applications est de fournir la résilience du système contrôlé par rapport à un dysfonctionnement du capteur éventuel. Alors que la fusion de capteurs atténue les bruits de mesure et certains événements de type défaut, il est également possible qu'un choix inapproprié à ce stade critique peut affecter considérablement la performance de l'usine.

La réalisation du système multi-capteurs via le réseau de communication partagé peut rendre la stratégie de commande tolérante aux défauts encore plus compliquée en raison des effets introduits par le réseau. Il est déjà indiqué dans les chapitres précédents que les retards introduits par le réseau sont très variables (sauf pour certains protocoles réseau avec des retards constants).

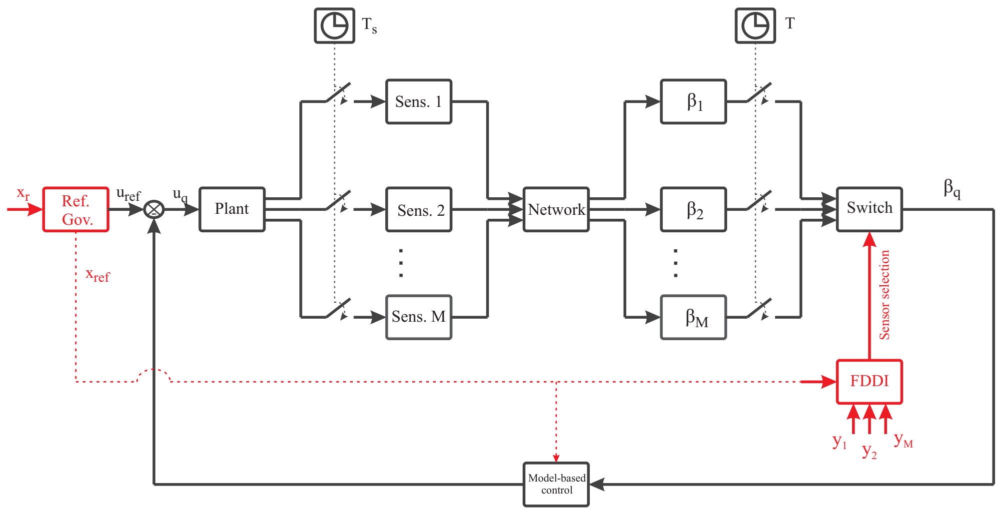


FIGURE 12: Système de commande multi-capteur (asservisement d'état)

Dans cette étude, nous considérons un NCS multi-capteur actif qui fournit une tolérance aux défauts en ce qui concerne les défauts de capteurs. L'objectif principal de ce chapitre est de concevoir un mécanisme qui est capable d'identifier les données, parmi les mesures redondantes disponibles, qui sont fournies par un capteur qui est affecté par un défaut. Ceci est réalisé par un mécanisme détection des défauts (FDI).

Considérons le système de commande représenté dans la Fig. 12, où le système est contrôlé par M capteurs redondants. La communication entre les capteurs et le correcteur est réalisée par un réseau partagé tandis que le correcteur colocalise avec l'actionneur. En raison des retards introduits par le réseau, il est possible que les capteurs aient un taux différent de transmission de données. En outre, il est supposé que les capteurs sont également soumis à des défauts occasionnels ou permanents.

Le processus est supposé être decrit en temps continu, modélisé par une équation différentielle linéaire de la forme suivante:

$$\dot{x}(t) = A_c x(t) + B_c u(t) + E_c \omega(t), \tag{11}$$

avec $x \in \mathbb{R}^n$ vecteur d'état, $u \in \mathbb{R}^m$ signal de commande et $\omega \in \mathbb{R}^p$ est un bruit de moyenne nulle. Le bruit de traitement est délimité par l'ensemble $\mathcal{W} \subset \mathbb{R}^p$ tel que $\omega \in \mathcal{W}$. Les matrices $A_c \in \mathbb{R}^{n\times n}$, $B_c \in \mathbb{R}^{n\times m}$ et $E_c \in \mathbb{R}^{n\times p}$ sont constantes.

Sans perte de généralité et en raison des notations simples, nous examinons le cas où tous les capteurs sont capables de mesurer le vecteur d'état complet.

Les mesures qui sont acquises par les capteurs sont transmises au dispositif de commande et mémorisées dans les tampons de réception. Chaque canal de l'asservisement est suivi par un tampon de réception qui est supposé être assez grand pour stocker les informations qui correspondent à une période d'échantillonnage, c'est-à-dire, le rejet de données à cause d'un débordement de tampon n'est pas traité. Tous les tampons sont échantillonnés périodiquement avec la période $T = \frac{T_s}{N}$, où $N \in \mathbb{Z}^+$ est suffisamment grand. Le correcteur est muni d'un mécanisme de commutation qui sélectionne un tampon pour le calcul du signal de commande à chaque instant entre les échantillonnages. Selon les informations disponibles, la décision qui sera utilisée par le tampon est faite par un mécanisme FDI qui supprime les informations en provenance des capteurs défectueux. En outre, le mécanisme de FDI signale également lorsque les mesures employées sont dépassées si le correcteur est capable de s'adapter à l'action de la commande en fonction des informations disponibles.

Un défaut peut officiellement être décrit comme une transition instantanée entre le mode de fonctionnement défectueux et le mode sain de fonctionnement. Pour la simplicité de la présentation nous considérons ci-après seulement les pannes totales des capteurs :

$$y_j[t_k] = x[t_k] + \eta_j[t_k] \xrightleftharpoons[RECOVERY]{FAULT} y_j[t_k] = 0 \cdot x[t_k] + \eta_j^F[t_k]. \tag{12}$$

Conclusions et perspectives

Deux méthodes basées sur des ensembles invariants pour la commande d'un système en temps discret avec des retards ont été examinées. La première s'appuie sur les ensembles invariants positifs, en particulier, les ensembles $D-$invariants qui sont robustes par rapport aux retards. Ce problème a également été abordé dans cette thèse par de notions de stabilité pour les équations de récurrence à retard à temps continu et à temps discret. Il a été montré que, à part leur application pratique, les ensembles $D-$invariants peuvent également apporter un nouvel éclairage sur la corrélation entre la stabilité asymptotique robuste qui est indépendante des retards. Par ailleurs, une solution alternative basée sur la factorisation est considérée et l'on peut représenter une direction de recherche possible afin d'obtenir la caractérisation d'un consigne plus souple. Cette idée ouvre également de nouvelles perspectives pour une meilleure gestion de la complexité des contraintes décrivant les ensembles invariants.

L'autre cadre considère une architecture avec plusieurs capteurs, éventuellement soumis aux défauts et aux retards. L' approche envisagée dans cette thèse repose sur des ensembles invariants robustes qui sont définis pour la dynamique nominale. Les retards et les défauts sont abordés par le correctuer qui est aussi robuste par rapport à la perturbation exogène. En ce qui concerne la partie d'atténuation de retard, nous soulignons qu'une telle action simple de commande est calculée et elle exploite la stratégie d'sous-échantillonnage. L'efficacité de la commande dépendra toutefois largement sur les contraintes imposées. En d'autres termes, comte tenu des informations de rétroaction mises à jour dans l'intervalle de temps admissible, on peut calculer des mesures de correction en cas d'erreur de prediction. Mais dans des applications réelles, la possibilité de corriger certaines erreurs de prévision dépendra largement des contraintes imposées au signal de commande. De l'autre côté, une unité de commande qui est capable de fournir des signaux de commande d'une amplitude importante permet également la compensation des retards plus grands. L'avantage le plus important d'une telle commande en comparaison avec les correcteurs robustes, c'est qu'elle peut assurer la stabilité d'un système, même en cas des systèmes qui peuvent être instables en présence d'une commande classique robuste.

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Notations:

$\mathbb{Z}$ - set of integers

$\mathbb{Z}_+$, $\mathbb{Z}^+$ - set of non negative and positive integers, respectively

$\mathbb{Z}_{[a,b]} = \{i \in \mathbb{Z} \ : \ a \leq i \leq b\}$ - closed interval of integers

$\mathbb{R}$ - set of real numbers

$\mathbb{R}_+$, $\mathbb{R}^+$ - set of non negative and positive real numbers, repsectively

$\mathbb{C}$ - set of complex numbers

$\mathbb{D}$ - open unit disc in the complex domain

$\partial\mathbb{D}$ - unit circle in the complex domain

$\mathcal{C}(\mathbb{R}_{[a,b]}, \mathbb{R}^n)$ - space of continuous vector functions on the interval $\mathbb{R}_{[a,b]}$

$\mathcal{K}$ - class of continuous, strictly increasing functions $f \ : \ \mathbb{R}_+ \to \mathbb{R}_+$ such that $f(0) = 0$

$\mathcal{K}_\infty$ - class of functions that belong to $\mathcal{K}$ and $\lim\limits_{x\to\infty} f(x) = \infty$

$\mathcal{KL}$ - class of functions $f \ : \ \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that for each fixed $k \in \mathbb{R}_+$, $f(\cdot, k) \in \mathcal{K}$ and for each fixed $x \in \mathbb{R}_+$ is decreasing and $\lim\limits_{k\to\infty} f(x, k) = 0$

$det(A)$ - determinant of the matrix A

$dim(\mathcal{B})$ - dimension of the space $\mathcal{B}$

$\rho(A) = \max\{|\lambda| \ : \ det(\lambda I - A) = 0\}$ - spectral radius of the matrix A

$\sigma(A) = \{\lambda \ : \ det(\lambda I - A) = 0\}$ - spectrum of the matrix A

$\sigma(A, B) = \{\lambda \in \mathbb{C} \ : \ \det(A - \lambda B) = 0\}$ - set of generalized eigenvalues

$A \otimes B$ - the Kronecker product of the matrices A and B

A^T and A^* - transpose and conjugate transpose of the matrix A, respectively

$\alpha_{n\times m}$ - matrix in $\mathbb{R}^{n\times m}$ with all entries equal to $\alpha \in \mathbb{R}$

I_n - identity matrix in $\mathbb{R}^{n\times n}$

$BlkDiag(Q, s)$ - block diagonal matrix with respect to $Q \in \mathbb{R}^{n\times n}$ and $s \in \mathbb{Z}^+$

$ConvHull(\mathcal{X})$ - convex hull of the set $\mathcal{X}$

$\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$ - p - norm of the vector $x \in \mathbb{C}^n$

$\mathbb{B}_\varepsilon^p(x) = \{y \ : \ \|x - y\|_p \leq \varepsilon\}$ - p-norm hypersphere centered at x

$\bigoplus\limits_{i=a}^{b} \mathcal{X}_i = \mathcal{X}_a \oplus \ldots \oplus \mathcal{X}_b$ - the Minkowski sum of the sets $\mathcal{X}_a, \ldots, \mathcal{X}_b$

$\bigoplus\limits_{i\in\mathcal{I}} \mathcal{X}_i$ - the Minkowski sum of sets determined by all $i \in \mathcal{I}$

$\mathcal{X} \times \mathcal{Y} = \{(x, y) \ : \ x \in \mathcal{X}$ and $y \in \mathcal{Y}\}$ - the Cartesian product of the sets $\mathcal{X}$ and $\mathcal{Y}$

$card\{\mathcal{X}\}$ - cardinality of the set $\mathcal{X}$

$int(\mathcal{X})$ and $ext(\mathcal{X})$ - interior and exterior of the set $\mathcal{X}$, respectively

List of abbreviations:

CAN - Control Area Network

CSMA/CD - Carrier Sense Multiple Access/Collision Detection

CSMA/AMP - Carrier Sense Multiple Access/Arbitration on Message Priority

DDE - Delay Difference Equation

dDDE - discrete-time Delay Difference Equation

cDDE - continuous-time Delay Difference Equation

DDI - Delay Difference Inclusion

FDDI - Fault and Delay Detection and Isolation

LMI - Linear Matrix Inequalities

LP - Linear Programming

LF - Lyapunov Function

LQR - Linear Quadratic Regulator

LTI - Linear Time-Invariant

MAC - Medium Access Protocol

MIP - Mixed-Integer Programming

MPC - Model Predictive Control

mRPI - minimal Robust Positively Invariant

mRDI - minimal Robust D-Invariant

NCS - Networked Control Systems

NP-hard - Non-Polynomial hard

QP - Quadratic Programming

RC - Reconfiguration Control

RPI - Robust Positively Invariant

RDI - Robust D-invariant

To Marija

“The Devil is in the details.”

Chapter 1

Introduction

THE problem treated in this thesis includes process regulation based on the feedback information affected by delay. This problem can be regarded in a general perspective of the fault tolerant control (FTC) design where delays are considered as a particular mode of performance degradation. The same problem can be also considered other way around. Namely, each FTC scheme requires some degree of redundancy in the installed components or subsystems (sensors, actuators, controllers). Larger number of nodes implies increased usage of communication resources in order to operate such a system. Due to bandwidth limitations and network congestion, data transmission between nodes may be compromised by delay or packet dropouts. This suggests that the analysis and control design for time-delay systems and the FTC design often appear together due to their interdependences.

Problems of the analysis and control design for time-delay systems and fault tolerant control (FTC) are largely treated in the literature, to mention only a few: Niculescu [2001], Richard [2003], Sipahi et al. [2011] (for a general overview on time-delay systems) and Blanke [2003], Zhang and Jiang [2008] (for a general overview on FTC design). In this study we provide an unified control framework in order to address the problem of delays appearing in a process regulation. For this purpose, we use networked control systems (NCS) as a modeling class. This particular class of dynamics provides general models that fit our objectives. Moreover, NCS as a popular research area nowadays (see Hespanha et al. [2007], Murray et al. [2003]), can also benefits from the results contained in this manuscript.

In general, FTC design approaches can be grouped into the passive and active methods. The passive methods considers control design that is robust with respect to a set of predefined faults. On the other side, an active scheme reacts to a detected fault and reconfigures the control actions so that the stability and the performances can be verified (see Stoican and Olaru [2013]). We can think in the same way the control design for delay attenuation. Namely, designing a robust controller in order to deal with delays is a classical problem in the control area (see Niculescu [2001]). Also, designing an active "delay tolerant control" also exists and the results are mainly relied on the predictor-based controllers (see e.g. Witrant et al. [2007]). In order to make these concepts clearer to the reader propose an analogy with the simple example of a cyclist. Imagine a road which consists of the fast lane in the middle and the slow and the emergency lanes from both sides. The objective of the cyclist is to keep his bicycle on the fast lane. Imagine that the cyclist closes his eyes periodically for a certain time interval. When cycling with his eyes shut, the cyclist can either hold the same direction as at the moment when he closed the eyes or he can estimate the trajectory on a finite horizon (based on the last available information) and keep performing some control action. The first scenario can be considered as the robust approach, while the second incorporates some active delay management. However, in the presence of disturbances (going back from a celebration for instance) no matter which "cycling approach" he employs, there is a big chance that he will (fortunately) find himself in the slow or the emergency lane, depending on the how long he kept his eyes closed. From this simple example we can notice an analogy with the FTC that is: the active fault (delay) management may provide better control performance and it can handle a wider range of faults (delays). However, in the presence of exogenous and unmeasured disturbance, prediction-based approach may not respond properly to the control requirements. We consider this and the similar problems throughout this thesis.

1.1 Objective of the thesis

There are two main directions followed in the thesis. The first one is to design an active control strategy, which is robust to the exogenous disturbance, in order to handle constraints and time-varying delays appearing in the control loop. The obtained control algorithm is also applied in a multi-sensor configuration. Thus control realization provides a general framework for dealing with sensor abrupt faults and network-induced delays in the sensor-to-controller communication channels. These results are implemented by using the set-theoretic framework, in particular *positively invariant sets*. The concept of positive invariance brings us to the second research direction considered the thesis where a strong invariance notion for discrete-time systems with delays is examined. This notion is known as the *D-invariance* and it is particularly useful in the model predictive control design for systems with large delays.

The objectives of this thesis are:

- To characterize $D-$invariance sets for the linear systems with additive disturbance and to provide a numerical method for their construction.
- To contribute to the establishment of necessary and sufficient conditions for $D-$invariance. There is a need in this sense because the existing conditions are not constructive, while the ones that are constructive are either necessary or sufficient.

- To provide a delay detection mechanism based on the reference governor design. Furthermore, to develop an active control method for compensation of network-induced delays which is robust with respect to bounded exogenous disturbance.

- To establish a multi-sensor control scheme with joint abrupt fault and delay tolerance capabilities thus generalizing the fault scenarious in the set-theoretic FTC methods.

1.2 Outline of the thesis and publications

The contents of the thesis are as follows:

Chapter 2: Networked control systems: Preliminaries and prerequisites In this chapter we consider the modeling for a continuous-time process which is controlled by a digital controller over a shared network. We start the discussion by a brief overview on various network protocols and their particularities with respect to delays and packet dropouts. Then, we regard discrete-time representation of a NCS which takes into consideration random, time-varying delays and packet dropouts. We also outline some theoretical results, along with the most recent developments in the analysis and the control design concerning NCS.

Chapter 3: Positive invariance for delay difference equations

The main objective of this chapter is to provide a support in terms of positively invariance sets for fault and delay detection algorithms. The second objective is to induce a general positive invariance notion for the NCS models. For this purpose, we provide an overview of different concepts of invariance for discrete-time systems with delays. Moreover, we detail some new insights on the existence and construction, in particular for $D-$invariant sets related to the linear delay-difference equations with additive disturbance. Furthermore, some alternative invariance solutions are discussed as well.

The results of this chapter can be found in:

- **Stanković, N.**, Olaru, S. and Niculescu, S.-I. (2011): "Further Remarks on Invariance Properties of Time-Delay Systems." In Proceedings of the ASME IDETC / MSNDC, Washington DC.

- **Stanković, N.**, Olaru, S. and Niculescu, S.-I. (2011): "Further Remarks on Invariance Properties of Time-Delay and Switching Systems." In Proceedings of the ICINCO, Noordwijkerhout.

- Olaru, S., **Stanković, N.**, Bitsoris, G. and Niculescu, S.-I. (2013): "Low Complexity Invariant Sets for Time-Delay Systems. A Set Factorization Approach." In the edited volume dedicated to Low Complexity Controllers for Time-Delay Systems, to appear in Springer 2014.

Chapter 4: Delay-difference equations. Stability and positive invariance

In this chapter the concept of strong stability for discrete-time and continuous-time delay-difference equations is considered. The importance of continuous-time delay-difference equations is the new insight that this class of system provides in analyzing the robust asymptotic stability and the existence of $D-$invariant sets.

The results of this chapter can be found in:

- **Stanković, N.**, Olaru, S. and Niculescu, S.-I. (2013): "Further Remarks on Asymptotic Stability and Set Invariance for Discrete-Time Delay-Difference Equations." Submitted to Automatica.
- **Stanković, N.**, Olaru, S. and Niculescu, S.-I. (2013): "On Stability of Discrete-Time Delay-Difference Equations for Arbitrary Delay Variations." In Proceedings of the IFAC TDS, Grenoble.

Chapter 5: Sensor-to-controller delays in NCS. Detection and control design

This chapter examines a control design for networked control systems with single sensor-to-controller communication channel. Random and time-varying delays, which can occur during data transmission through this channel, are regarded as faults. We design a set-based delay detection mechanism, which in general can be regarded as a fault detection and identification mechanism. Furthermore, a model-based controller with a compensation block which is capable to correct the induced tracking error is designed as well.

The results of this chapter can be found in:

- **Stanković, N.**, Stoican, F., Olaru, S. and Niculescu, S.-I. (2012): "Reference governor design with guarantees of detection for delay variation." In Proceedings of the IFAC TDS, Boston.
- **Stanković, N.**, Olaru, S. and Niculescu, S.-I. (2012): "Set-based detection and isolation of inter-sampled delays and packet dropouts in networked control." Lecture Notes in Artificial Intelligence, Springer.

Chapter 6: Multi-sensor NCS with tolerance to abrupt sensor faults

The main objective in this chapter is to design a mechanism which is capable to identify data, among available redundant measurements, which are provided by a sensor that is affected by an abrupt fault. Beside the abrupt faults, performance of the closed-loop dynamics can also be degraded by delayed data transmission. Therefore, we combine the model-based controller with active delay compensation in view of a fault detection and identification mechanism.

The results of this chapter can be found in:

- **Stanković, N.**, Olaru, S. and Niculescu, S.-I. (2013): "Multi-sensor networked control system with fault tolerance guarantees." Submitted to International Journal of Adaptive Control and Signal Processing.

- **Stanković, N.**, Olaru, S. and Niculescu, S.-I. (2014): "A fault and delay tolerant multi-sensor control scheme." Submitted to IFAC World Congress.

Chapter 7: Conclusions and further directions In the last chapter conclusions are presented. Extensions and open problems are also discussed.

Chapter 2

Networked control systems: Preliminaries and prerequisites

A control system realization that is supposed to include several (possibly many) nodes (interconnected subsystems) is often carried out in practice via shared network. This way of inter-connection between sensors, actuators and controllers brings numerous benefits such as reduced system wiring, increased flexibility and lower maintenance cost. However, transferring information in packets, which is a particularity of the NCSs, introduces some challenges that are usually not considered in the classical control theory. For instance, due to a packet size constraints, only limited information can be transferred by a single packet. Hence, a larger amount of data needs to be broken into multiple packets in order to be transmitted. Such bandwidth limitations represent a major difference from the conventional sampled-data systems where it is assumed that data are delivered at the same time. Another important issue related to the NCSs are the packet dropouts (or data rejection). Namely, due to unreliable links (typical for the wireless networks), it may happen that some packet are lost during the transmission. Moreover, for the real time control applications it is often advantageous to use only up-to-date information if and when they are available. Therefore, packet dropouts can also occur due to rejection of outdated information when up-to-date data become available. Communication over shared network can be source of delays too. In particular, the main source of delays in networked control systems represents time it takes for a node to access the network in order to transmit data. Delays can also be initiated by a time that is needed for the control computation or the data transmission through the network. All these network-induced effect need to be taken into consideration carefully because they can significantly degrade performance of the closed-loop dynamics.

In this chapter we consider the modeling for a continuous-time process which is controlled by a digital controller over a shared network. We start the discussion by a brief overview on various network protocols and their particularities with respect to delays and packet dropouts. In Section 2.2 we regard discrete-time representation of a NCS which takes into consideration random, time-varying delays and packet dropouts. Section 2.3 outlines the most relevant classical theoretical results, along with the most recent developments in the analysis and the control design concerning NCS. The chapter ends with a recapitulation of the presented notions and a reference overview of the related results.

2.1 A brief introduction to control networks

Depending on the different communication layers in a complex system, we may differ *data networks* and *control networks*. Data networks are characterized by large data packets and relatively infrequent transmission. On the other side, control networks are required to meet more strict specifications, imposed by time-critical applications (see Koren et al. [1996]) such as: frequent exchange of small data packets, guaranteed transmission or bounded time-delays. Some networks such as Ethernet, may be used as data and control network. However, in this study we consider only the control networks and their induced effects with respect to closed-loop system performance.

With distributed control systems, control functions can be moved out of central units into controllers located near the controlled devices (see Koren et al. [1996]). Moreover, such an architecture provides modularity in design, flexibility with respect to additional functionality to the system and better fault diagnostics and maintenance. The key component that facilitates implementation of the distributed control systems is a shared network. Moreover, networks can also decrease the wiring in the complex dynamical systems such as airplanes (see Eccles [1998]) and thus reduce the overall cost. This way of communication between different nodes or sub-systems obviously brings in many advantages when compared to the classical, point-to-point communication architecture.

Control implementation via network may also have a negative impact on performance of a closed loop system due to access delay, transmission time, response time, message delay, message collisions (percentage of collisions), message throughput (percentage of discarded packets), packet size, network congestion and determinism boundaries (see Lian et al. [2001]). Moreover, these network-induced effects are mutually dependent. For instance, relatively large data packets or a high frequency sampling require increased network utilization. Consequently, the network is subject to larger delays due to increased message collisions. This may also cause increased data queuing in the output buffers and packets dropout. Such a degraded network functioning may deteriorate response of the closed-loop dynamics, even lead to instability. Therefore, one of the most important tasks in NCS design is to choose suitable control network that is capable to meet a control performance specification.

Some control networks are deterministic in the sense that they induce constant or predictable delays. On the other hand, there are also networks that cause random and varying delay that, for example, depends on the network congestion. This is a direct consequence of the Medium Access Control (MAC) sublayer protocol. Namely, MAC protocol regulates access to the network for each node. Such decision can be taken based on data priority assignment, scheduling or it can be stochastic. Consequently, the corresponding closed-loop dynamics model is constant or time-variant. From this fact, one can conclude that complexity of the analysis and the control design are considerably affected by a chosen control network. In this study, we consider three popular industrial network standards: Ethernet, ControlNet and Control Area Network (CAN) and their specifications regarding time-delay and packet dropout.

2.1.1 Timing analysis and network-induced delays

Analysis of network-induced delays is particularly important for processes which are characterized by very fast dynamics such as, for instance, flight control (see Ray [1987]), while these delays often are considered to be insignificant in relatively slow processes, such as those encountered in manufacturing (see Ray [1989]).

NCSs lies at the intersection of control and communication theories (see Hespanha et al. [2007]). For this reason it may be appropriate to clarify the notion of time-delays with respect to both fields. Namely, communication theory considers only delays with respect to successfully transmitted data. Those delays are primarily caused by queuing and serial transmission of bits, while the corrupted or deleted messages are not taken into account. However, in control theory, delays are related to the question: *How old are the data that are currently used* (see Ray [1989])? One can notice that these two notions are significantly different if some packets are rejected. In order to better understand delay appearance in a network, a general timing diagram is outlined and detailed in this subsection. A schematic representation, which is outlined in Lian et al. [2001], is shown on Fig. 2.1.

Absolute time-delay for a message transmission, T_{delay}, is defined as the time it takes for a node to transmit (deliver) the entire message to its destination. This delay can be determined by:

$$T_{delay} = T_{pre} + T_{wait} + T_{trans} + T_{post}, \tag{2.1}$$

where T_{pre} is the *preprocessing* delay, T_{wait} is the *waiting* delay, T_{trans} is the *transmission* delay and T_{post} is the *post-processing* delay.

The preprocessing delay is determined as $T_{pre} = T_{scom} + T_{cod}$, where T_{scom} is the computation time of the source node and T_{cod} is the encoding time of the message that should be sent. After the preprocessing period, the message is transferred to the corresponding output buffer, from where it should be sent to the destination node. However, if there are already other messages that are waiting to be transmitted, the data will queue until all previous messages are transmitted. The queuing time of a message is denoted by T_{queue}. Moreover, the queuing time can be also zero, since some protocols send only the last acquired data while all unsent outdated measurements are discarded. When the

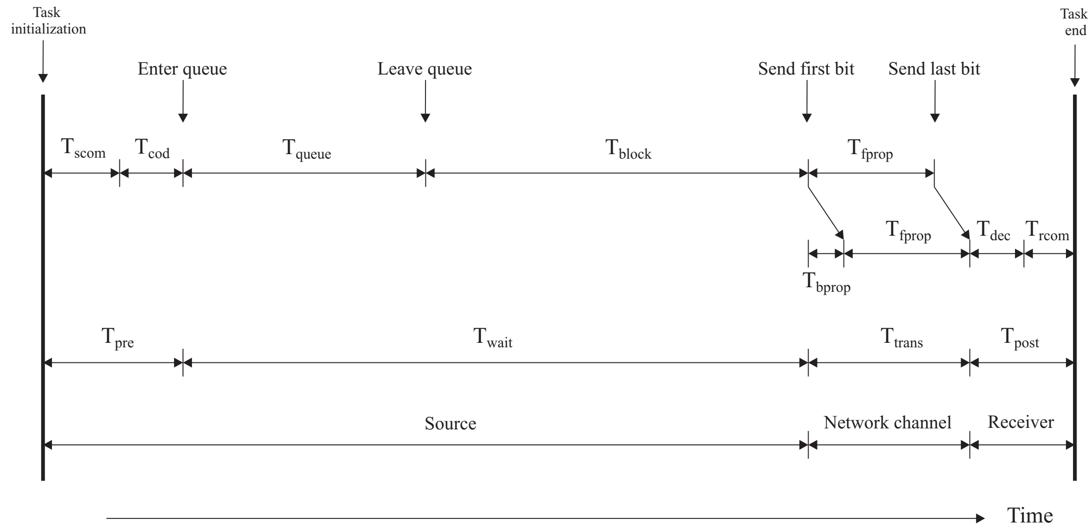


FIGURE 2.1: A timing diagram in NCS

message is ready to be transmitted, the corresponding node requires permission from the MAC protocol to access the network. If the network is occupied, then the transmission is blocked until the network is idle. The delay, which is induced by blocking the message transmission due to network utilization, is denoted by T_{block}. $T_{wait} = T_{queue} + T_{block}$ determines the waiting delay. The transmitted first bit of the message is delivered to its destination after T_{bprop}, where T_{bprop} determines the propagation time for one bit and it depends on the physical length of the network. Time it takes to transmit the entire message T_{fprop}, determines the propagation time for the entire message (frame), where $T_{trans} = T_{bprop} + T_{fprop}$. The delivered message is decoded and post-processed. The corresponding delays in this case are denoted by T_{dec} and T_{rcomp}, respectively, while $T_{post} = T_{dec} + T_{rcom}$.

For a specific control network, T_{pre}, T_{post} and T_{trans} can be considered constant compared to the waiting delay (see Lian et al. [2001]), i.e., T_{delay} can be regarded only with respect to T_{wait}. This delay, also denoted by *access-time delay*, has been identified as the main source of delay in NCS (see Lin and Antsaklis [2005]) and it differs with respect to different MAC protocols. In order to provide better insight in network-induced delay with respect to a specific MAC protocol, in the subsequent subsections we consider basic functioning of three popular industrial control networks.

2.1.2 Ethernet (CSMA/CD)

Ethernet with *Carrier Sense Multiple Access / Collision Detection* (CSMA/CD) is an industrial control network specified in the IEEE 802.3 standard. CSMA/CD is the mechanism for resolving contentions on the communication medium. Namely, before it transmits a message, each node listens to the network. If the network is idle, the node transmits bit by bit until the whole message is delivered to its destination. If however, the network is busy, the message transmission is blocked and put on hold. Sometimes it may happen that two or more nodes listen to the network in order to gain access, so they can start transmission simultaneously. The messages of these nodes collide, and the transmitted information gets corrupted. When such a collision is detected, concerned nodes stop their transmissions and wait random time before retrying the transmission. This random time is determined by the *binary exponential algorithm* (for more details see Lian et al. [2001]). After certain numbers of collisions, node stops attempting to transmit data and reports failure.

The Ethernet with CSMA/CD utilizes a simple algorithm for network access which does not additionally exploit the available network resources. This protocol can carry larger data packets, so it is equally used in control and data networks. The main advantage of this network access algorithm is that it has almost no delays at low network loads (see Wheels [1993]). However, Ethernet is a non-deterministic protocol in the sense that it induces random time-delays. Moreover, it does not support any message prioritization. At high network loads, message collisions become a major issue since they significantly increase delays, data rejections and the overall performance of the network and, consequently, the controlled system. Another characteristic of the Ethernet protocol is that it uses relatively larger data packets to transmit even small amount of information. This is due to the minimal data packets requirements by the standard.

2.1.3 ControlNet

ControlNet is a *Token-passing* bus, specified in the IEEE 802.4 standard. It is a deterministic network protocol since the transmission time of a message is bounded and determined by the token rotation. All nodes at the network are arranged into a logical ring. Each node posses also the address of its logical predecessor and successor. The transmission is guaranteed for a node that holds the token. Each node transmits the data either until the whole message is transmitted or the time it held the token reaches a limit (see Lian et al. [2001]). At the end of its transmission, the node passes the token to its logical successor. Since only one node is allowed to transmit data at a given moment, there is no message collision and the protocol guarantees the maximum time between network access for each node.

ControlNet protocol is a deterministic protocol, i.e., it guarantees constant time delays. It also provides good efficiency at high network loads. In contrast to the token passing networks where the nodes are arranged in a physical ring, ControlNet can dynamically add or remove nodes from the bus. However, despite good performance at lower network loads, ControlNet cannot achieve performance of contention networks such as Ethernet. Moreover, in the case where there are many nodes connected to the token passing bus, a significant network resource is used in passing the token between the nodes, i.e., additional communication bandwidth is required to gain access to the network.

2.1.4 DeviceNet (CSMA/AMP)

DeviceNet is a controller area network (CAN) with *Carrier Sense Multiple Access / Arbitration on Message Priority* (CSMA/AMP) medium access protocol. CAN protocol is defined by the $ISO11898-1$ international standard. It was primarily developed for the automotive industry, while today it can be found in a broad area of applications. The network is optimized for short messages and it is primarily used in device-level manufacturing applications. The medium access of simultaneous transmissions is resolved by message priority assignment. Namely, each node transmits a bit stream which is synchronized at the starting bit. The arbitration is performed bit-wise in the way that a node which transmits the logic zero is dominant over a node which transmits the logic one. When two or several nodes attempt to transmit data simultaneously, they first continue to send messages and then listen to the network. If one of them receives a different bit from the one it sends out, it loses the right to continue transmission and it switches into the receiving mode (see Lian et al. [2001]). Therefore, the ongoing transmission is never corrupted. DeviceNet, as other CAN networks, can work in broadcast or in multicast mode of operation. Namely, in broadcast mode, transmitter sends out a message that is received by all nodes. The decision is on each node whether it accept the received message or not. In the multicast mode, messages are only transmitted to a pre-defined group of nodes, but not all of them.

DeviceNet is a relatively low cost network and it is a deterministic protocol optimized for short messages. Moreover, it can alway guarantee the transmission time for high priority messages. On the other side, DeviceNet has a slow data rate due to the bit-wise arbitration, i.e., an additional network resource is utilized in order to decide which one of the conflicting nodes has priority for transmission. Also, the bit synchronization requirement limits the maximum length of a DeviceNet network. DeviceNet is not designed for large data transmission.

For more details on experimental results on network-induced delays for the presented protocols, we refer to Lian et al. [2001] and Nilsson [1998].

2.2 Modeling of NCS

In general, networked control systems are represented by several control loops effectuated over the same communication network (see Hespanha et al. [2007]). However, an often used architecture for the theoretical consideration is the single feedback loop which is depicted on the Fig. 2.2. Even though this scheme is considerably simpler than the

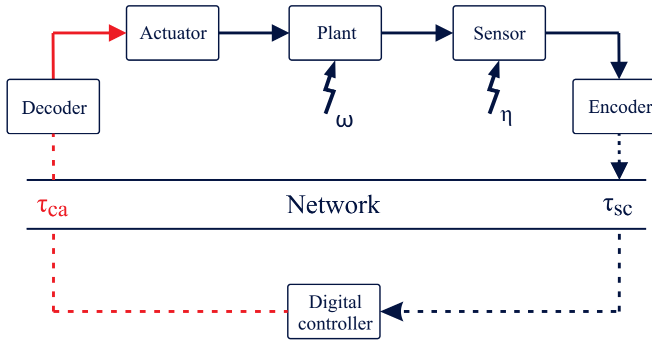


FIGURE 2.2: Single feedback loop networked control system

general NCS architecture, it still captures the main important issues initiated by shared network which are of the main interest in this work.

Briefly, in order to control a continuous-time plant via shared network the output signal must be sampled and encoded into a digital format. Obtained data are transmitted via network to the digital controller. Digital control, transmitted to the actuator also via network, is decoded and transformed into a continuous-time signal by a hold unit. Furthermore, the process and the measured output are usually affected by disturbances (notice ω and η on Fig. 2.2).

Let us consider a continuous-time plant which is represented by the following linear, time-invariant differential equation:

$$\dot{x}(t) = A_c x(t) + B_c u(t) + E_c \omega(t), \tag{2.2}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control input and $\omega \in \mathbb{R}^p$ is the process additive disturbance. It is assumed that $\omega \in \mathcal{W} \subset \mathbb{R}^p$, where $\mathcal{W}$ is a compact and convex set[1]. Matrices $A_c \in \mathbb{R}^{n\times n}$, $B_c \in \mathbb{R}^{n\times m}$ and $E_c \in \mathbb{R}^{n\times p}$ are constant.

Let assume that all network-induced delays between the sensor and the actuator can be neglected, i.e., measurements and control signals are instantaneously transmitted through the network. For the constant $T_s = t_{k+1} - t_k$ and by using the zero-order hold sampling, the discrete-time representation of (2.2) is given as:

$$x[t_{k+1}] = Ax[t_k] + Bu[t_k] + E\omega[t_k]^{2}, \tag{2.3}$$

$$A = e^{A_c T_s}, \quad B = \int_0^{T_s} e^{A_c\zeta} d\zeta B_c, \quad E = \int_0^{T_s} e^{A_c\zeta} d\zeta E_c. \tag{2.4}$$

[1]For the definitions on compactness and convexity see Section 3.1

[2]The square brackets are used in order to point out the discrete-time nature of the variables.

In this study, the matrix exponential (2.4) is approximated by using series expansion such that:

$$A = I_n + A\Psi, \quad B = \Psi B_c, \quad E = \Psi E_c,$$

where

$$\Psi = \int_0^{T_s} e^{A_c \zeta} d\zeta = I_n T_s + \frac{A_c T_s^2}{2!} + \frac{A_c^2 T_s^3}{3!} + \ldots + \frac{A_c^i T_s^{i+1}}{(i+1)!} + \ldots.$$

Even though is not the most reliable way of computing the matrix exponential, this method meets most of our requirements for simple examples that are considered throughout this thesis. Other ways of computing the matrix exponential can be found in the survey paper of Moler and Van Loan [2003].

If there are no delays induced by the network, one can notice that the control action $u[t_k]$ is constant over the entire sampling interval, i.e., $\forall t \in \mathbb{R}_{[t_k, t_{k+1})}$. Moreover, in (2.3) it is assumed that the exogenous disturbance signal ω is constant between two consecutive samplings. This assumption will be relaxed later.

Hypothesis 2.1. All internal clocks are synchronized.

Regarding the discrete-time model (2.3), digital control signal $u \ : \ \mathbb{R}^n \to \mathbb{R}^m$ is given by the following piecewise-constant function:

$$u(t) = u[t_k], \quad \forall t \in \mathbb{R}_{[t_k, t_{k+1})}. \tag{2.5}$$

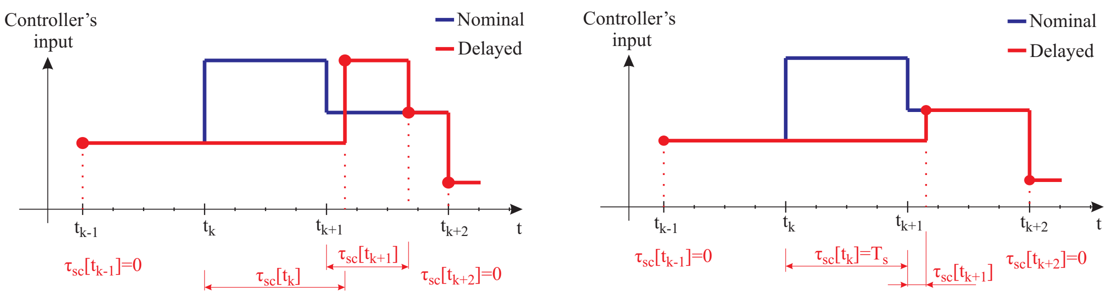
Consider the case when communication between nodes is affected by delays. Namely, at $t = t_k$, $k \in \mathbb{Z}_+$, each node (sensor and controller) requires permission to access the network in order to transmit data. Depending on network congestion at the moment and specific MAC sublayer protocol, we may have that either one node transmits data while the others have to wait until the network is idle (see ControlNet or DeviceNet in the previous section), or both nodes have to back off and wait random time (see Ethernet in the previous section). Without loss of generality, all particular network-induced delays, that have been outlined in Subsection 2.1.1, are considered as *sensor-to-controller* and *controller-to-actuator* delays (see τ_{sc} and τ_{ca} on Fig. 2.2, respectively).

Introduce the following piecewise-constant delay function:

$$\tau \ : \ \mathbb{R}_+ \to \mathbb{R}_+, \quad \tau(t) = \tau[t_k], \quad \forall t \in \mathbb{R}_{[t_k, t_{k+1})}, \tag{2.6}$$

where $\tau = \tau_{sc} + \tau_{ca}$ denotes a total delay such that $\tau[t_k] \in \mathbb{R}_{[0, \tau_{max}]}$, $\forall k \in \mathbb{Z}_+$, with $\tau_{max} < \infty$.

In the real time control applications, it is often advantageous to transmit only the most recent data if and when they are available (see Zhang et al. [2001]). In other words,



(A) Data queue (B) Data rejection

FIGURE 2.3: Timing diagram for data queue and data rejection

there is no cumulation of messages in the output buffer of a node (see T_{queue} in Subsection 2.1.1). Instead, only the most recent information is transmitted, while the unsent messages are rejected. In this way, receiving nodes are provided by an up-to-date information when it is available. Moreover, an additional network loading is avoided. A schematic comparison is provided on Fig. 2.3.

Hypothesis 2.2. Each node transmits only the most recent data.

Remark 2.1. For network-induced delays that are larger than a sampling period Hypothesis 2.2 does not correspond to any of the previously discussed control networks. However, it will be often use throughout this study. It is therefore important to stress that eventual practical implementation of the results that are provided in this work may require a control network with some non-standard MAC sublayer protocol for delays larger than a sampling period. Hypothesis 2.2 obviously holds if network-induced delays are smaller than a sampling period.

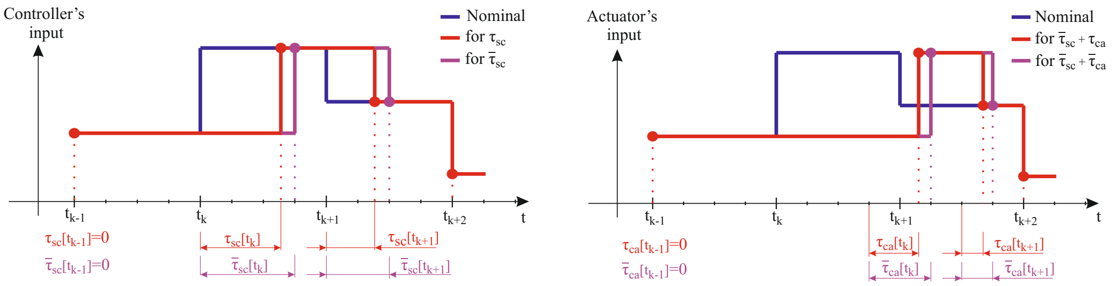
Let assume that the sensor is time-driven[3] while the controller and the actuator are event-driven[4] nodes. For a random delay variation, the possible number of active control inputs during one sampling period is a variable that depends on delay parameter. As an example of the timing behavior in NCS, we refer to Fig. 2.4 where the controller is assumed to be an identity map.

Taking into consideration packet dropouts due to data rejection, a discrete-time model of (2.2) which is controlled via network is provided by Cloosterman et al. [2009].

Proposition 2.1 (Cloosterman et al. [2009]). *Let* $\tau[t_{k+j-d_m}] \in \mathbb{R}_{[0,\tau_{max}]}$, $\forall j \in \mathbb{Z}_{[0,d_m]}$. *The control action on the sampling interval* $\mathbb{R}_{[t_k,t_{k+1})}$ *is given by*

$$u(t) = u[t_{k+j-d_m}], \quad \forall t \in \mathbb{R}_{[t_k+s_j[t_k],t_k+s_{j+1}[t_k])}, \tag{2.7}$$

[3]The output is measured periodically with each pulse of the clock.
[4]All signals are implemented as soon as they are received.



(A) Controller's input (B) Actuator's input

FIGURE 2.4: Signal propagation through the control loop affected by delays

where $d_m = \left\lceil \frac{\tau_{max}}{T_s} \right\rceil$ *and*

$$\begin{aligned} s_j[t_k] = \min\{ & \max\{0, \tau[t_{k+j-d_m}] + (j - d_m)T_s\}, \\ & \max\{0, \tau[t_{k+j-d_m+1}] + (j - d_m + 1)T_s\}, \dots, \\ & \max\{0, \tau[t_k]\}, T_s\}, \end{aligned} \tag{2.8}$$

with $s_j[t_k] \leq s_{j+1}[t_k]$, $j \in \mathbb{Z}_{[0,d_m]}$ *and* $s_0[t_k] = 0$, $s_{d_m+1}[t_k] = T_s$.

The corresponding state vector is determined according to the following difference equation:

$$x[t_{k+1}] = Ax[t_k] + \sum_{j=0}^{d_m} B_{d_m-j}(\tau_k)u[t_{k+j-d_m}] + E\omega[t_k] \tag{2.9}$$

where $B_{d_m-j}(\tau_k) = \int\limits_{T_s-s_{j+1}[t_k]}^{T_s-s_j[t_k]} e^{A_c\zeta}d\zeta B_c$ *and* $\tau_k = \begin{bmatrix} \tau[t_k] & \tau[t_{k-1}] & \dots & \tau[t_{k-d_m}] \end{bmatrix}^T$.

Proof. See Cloosterman et al. [2009]. □

Remark 2.2. In this thesis, discrete-time models with delays (as for instance (2.9)) are used to analyze the behavior of the closed-loop systems controlled over the network. However, problem of network-induced delays can be also regarded in the context of delay-differential equations with piecewise control and delay at the input (see Răsvan and Popescu [2001]).

Since $\tau[t_{k+j-d_m}] \in \mathbb{R}_{[0,\tau_{max}]}$, $j \in \mathbb{Z}_{[0,d_m]}$, one can notice that (2.9) is a time-variant dynamics. Stability analysis of such dynamics is not simple and it is usually based on finding a common Lyapunov function for all system representations defined over an infinite set (for $\forall\tau[t_{k+j-d_m}] \in \mathbb{R}_{[0,\tau_{max}]}$, $j \in \mathbb{Z}_{[0,d_m]}$). In order to practically determine a common Lyapunov function for such a system, one can use a finite grid of the delay range, which leads to a finite set of linear matrix inequalities (see Zhang et al. [2001]). It is also important to stress here that, if there exists, a common Lyapunov function represents only a sufficient stability condition, and as such it may represent a conservative result. An overview of the most relevant theoretical results concerning stability analysis of time-variant linear systems is outlined in the subsequent section.

An approach that provides a finite approximation of (2.9) is proposed by Lin and Antsaklis [2005]. In this work, the authors proposed a NCS scheme with receiving buffers at the controller's and at the actuator's site which are read periodically at a higher frequency then the sensor's sampling frequency T_s. Namely, buffers are assumed to be sampled with the period T, such that $T_s = NT$, with sufficiently large $N \in \mathbb{Z}^+$[5]. A NCS model obtained in this way has a finite number of different configurations, i.e., it is represented by a switching dynamics. This is formally outlined in the following proposition.

Proposition 2.2. *Let $\tau = \tau_{sc} + \tau_{ca} \in \mathbb{R}_+$. Denote by $\bar{\tau} = \bar{\tau}_{sc} + \bar{\tau}_{ca}$, where*

$$\bar{\tau}_{sc} = \left\lceil \frac{\tau_{sc} N}{T_s} \right\rceil T, \quad \bar{\tau}_{ca} = \left\lceil \frac{\tau_{ca} N}{T_s} \right\rceil T, \quad T_s = NT.$$

For $\tau[t_{k+j-d_m}] \in \mathbb{R}_{[0,\tau_{max}]}$, $\forall j \in \mathbb{Z}_{[0,d_m]}$, $d_m = \left\lceil \frac{\tau_{max}}{T_s} \right\rceil$, the control action on the sampling interval $\mathbb{R}_{[t_k,t_{k+1})}$ is given by

$$u(t) = u[t_{k+j-d_m}], \quad \forall t \in \mathbb{R}_{[t_k+\bar{s}_j[t_k],t_k+\bar{s}_{j+1}[t_k])}, \tag{2.10}$$

where

$$\begin{aligned} \bar{s}_j[t_k] = \min\{ & \max\{0, \bar{\tau}[t_{k+j-d_m}] + (j - d_m)T_s\}, \\ & \max\{0, \bar{\tau}[t_{k+j-d_m+1}] + (j - d_m + 1)T_s\}, \dots, \\ & \max\{0, \bar{\tau}[t_k]\}, T_s\}, \end{aligned} \tag{2.11}$$

with $\bar{s}_j[t_k] \leq \bar{s}_{j+1}[t_k]$, $j \in \mathbb{Z}_{[0,d_m]}$ and $\bar{s}_0[t_k] = 0$, $\bar{s}_{d_m+1}[t_k] = T_s$.

The corresponding state vector is determined according to the following difference equation:

$$x[t_{k+1}] = Ax[t_k] + \sum_{j=0}^{d_m} B_{d_m-j}(\bar{\tau}_k)u[t_{k+j-d_m}] + E\omega[t_k] \tag{2.12}$$

where $B_{d_m-j}(\bar{\tau}_k) = \int_{T_s-\bar{s}_{j+1}[t_k]}^{T_s-\bar{s}_j[t_k]} e^{A_c\zeta} d\zeta B_c$ and $\bar{\tau}_k = \begin{bmatrix} \bar{\tau}[t_k] & \bar{\tau}[t_{k-1}] & \dots & \bar{\tau}[t_{k-d_m}] \end{bmatrix}^T$.

Proof. Results provided in this proposition are obtained based on Proposition 2.1. For more details we refer to Cloosterman et al. [2009]. □

[5] This smaller sampling period will be often referred to as the inter-sampling period.

Remark 2.3. Using buffers in the control loop results in retarded controller's and actuator's reaction with respect to the case when these nodes are solely event-driven. In other words, all updates are taken into consideration only when the buffers are read. However, by choosing a sufficiently large N (N is a design parameter), buffers are read with a higher frequency, which causes faster controller's and actuators reaction. On the other side, a relatively smaller N is a trade-off associated with a simpler model (2.12) which consequently has less possible configurations.

The main benefit of introducing buffers in the control loop is that we obtain dynamics with a finite number of different modes. In other words, time-variant model (2.9) is replaced by the linear switching dynamics (2.12). This brings in several advantages from the stability analysis point of view as well. The first one is that the model (2.12) already represents an approximation of the linear time-variant dynamics (2.9) on a finite grid, thus no further approximations are required in order to numerically compute a common Lyapunov function. There also exists a necessary and sufficient stability condition which is based on the *joint spectral radius* computation. However, verifying such a condition remains a numerically complex problem. More details on these well-known theoretical results are provided in the subsequent section and the references therein.

Regarding sensor-to-controller and controller-to-actuator delays, it can be noticed that their 'nature' could be significantly different from the control design point of view. Namely, since sensor-to-controller delays precede the control action, one can conclude that it is possible to design a controller such that these delays are compensated (see Zhang et al. [2001]). On the other side, compensation of the controller-to-actuator delays is only possible if such delays are known in advance to the controller. Even though it is possible to know delays in advance for some network protocols (the one with constant delays), in general, such an assumption would be conservative.

Taking into consideration differences between τ_{sc} and τ_{ca}, we can discern two main directions in order to tackle network-induced delays. The first direction represents the robust control approach with respect to delay parameter. The main objective according to this approach would be to design a control action which guarantees stability of the closed-loop system for any $\tau[t_{k+j-d_m}] \in \mathbb{R}_{[0,\tau_{max}]}$, $j \in \mathbb{Z}_{[0,d_m]}$. Such an approach typically utilizes Lyapunov stability method and it is largely treated in literature (see e.g. Cloosterman et al. [2009], Lombardi et al. [2010] and Lin and Antsaklis [2005]). The second direction, which is less examined in the literature, is to consider an active strategy with delay detection and control reconfiguration in order to achieve delay compensation. Such an approach can be regarded as a *fault tolerant control*-type strategy with the fault detection and the control reconfiguration mechanisms so that the stability and the performances can be verified when a fault (in this case delay) occurs. One can notice that such a control action would only apply to sensor-to-controller delays. A specific NCS architecture that corresponds to this case is depicted on Fig. 2.5 where the controller is collocated with the actuator.

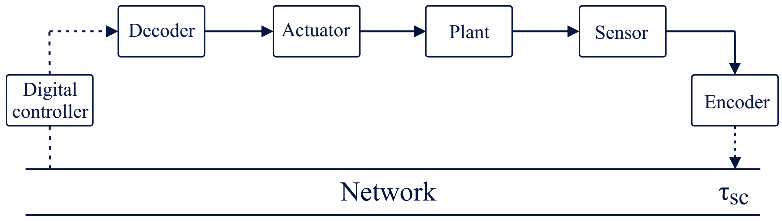


FIGURE 2.5: Single feedback NCS with controller collocated with the plant

The fact that the controller is collocated with the actuator results in a simpler discrete-time representation than the general one outlined in Proposition 2.1. Since $\tau = \tau_{sc}$, state vector of the NCS depicted on Fig. 2.5 is determined according to the following proposition.

Proposition 2.3. *Let $\tau[t_k] \in \mathbb{R}_{(0,\tau_{max}]}$. Then, $\tau[t_k] = (d[t_k]-1)T_s + \tau_{is}[t_k]$ where*

$$\tau_{is} \in \mathbb{R}_{[0,T_s)}, \quad d = \left\lceil \frac{\tau}{T_s} \right\rceil. \tag{2.13}$$

The corresponding state vector is determined according to the following difference equation:

$$x[t_{k+1}] = Ax[t_k] + B_0(\tau_{is})u[t_k] + B_d(\tau_{is})u[t_{k-d[t_k]}] + E\omega[t_k], \tag{2.14}$$

where $B_0(\tau_{is}) = \int\limits_0^{T_s-\tau_{is}[t_k]} e^{A_c\zeta} d\zeta B_c$ *and* $B_d(\tau_{is}) = \int\limits_{T_s-\tau_{is}[t_k]}^{T_s} e^{A_c\zeta} d\zeta B_c$.

The proof of the proposition is straightforward and it is ommited here. We refer to τ_{is} as the inter-sampling delay while $d[t_k]-1$ determines the number of dropouts (rejected packets) since the last control update.

Remark 2.4. Since there is one node in the control loop that transmits data over to the network, there are mostly two active control inputs per sampling period (see Fig. 2.6).

Difference equation (2.14) can be simplified in the same manner as that was done in Proposition 2.2 for the general NCS model. Namely, introducing a receiving buffer at the controller's input which is sampled periodically with the interval T, the state vector can be determined according to the following proposition which is reported without proof.

Proposition 2.4. *Let $\tau[t_k] \in \mathbb{R}_{(0,\tau_{max}]}$. Denote by $\bar{\tau}[t_k] = (d[t_k]-1)T_s + \bar{\tau}_{is}[t_k]T$, where*

$$\bar{\tau}_{is} \in \mathbb{Z}_{[0,N-1]}, \quad \bar{\tau}_{is} = \left\lceil \frac{\big(\tau - (d-1)T_s\big)N}{T_s} \right\rceil, \quad d = \left\lceil \frac{\tau}{T_s} \right\rceil, \quad T_s = NT. \tag{2.15}$$

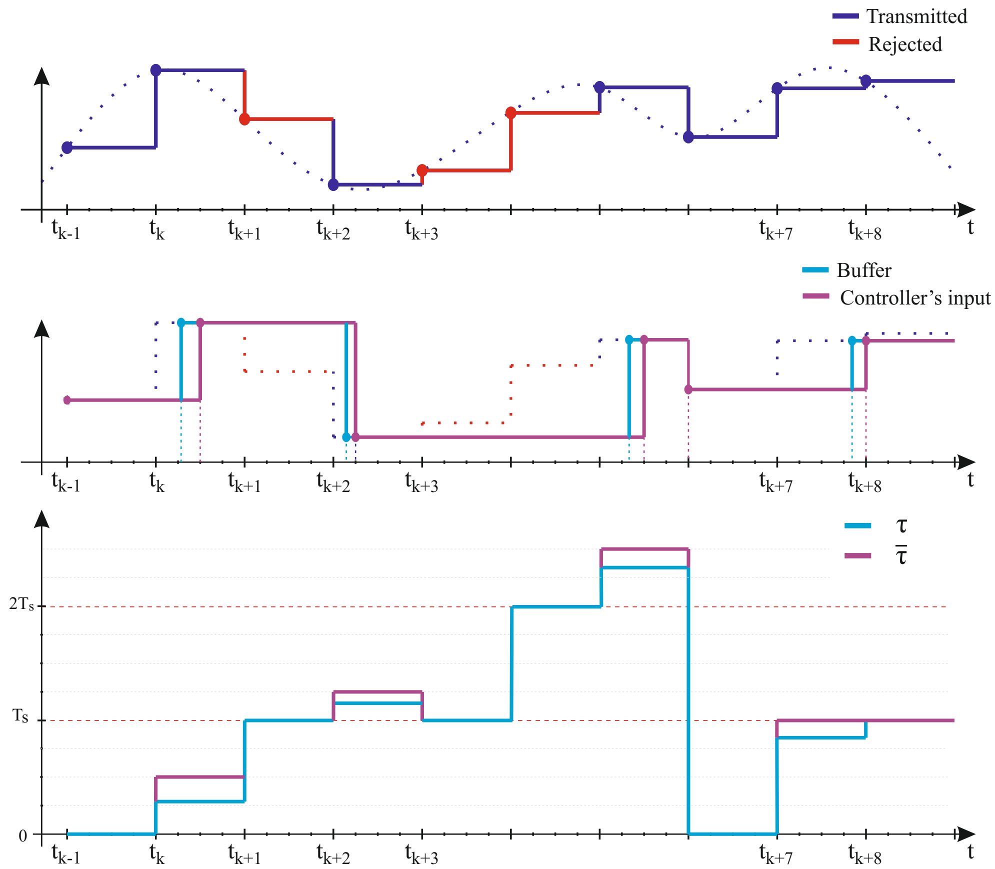


FIGURE 2.6: Sensor-to-controller delays

Then, the corresponding state vector is determined according to the following difference equation:

$$x[t_{k+1}] = Ax[t_k] + B_0(\bar{\tau}_{is})u[t_k] + B_d(\bar{\tau}_{is})u[t_{k-d[t_k]}] + E\omega[t_k] \tag{2.16}$$

where $B_0(\bar{\tau}_{is}) = \int_0^{T_s - \bar{\tau}_{is}[t_k]T} e^{A_c\zeta} d\zeta B_c$ *and* $B_d(\bar{\tau}_{is}) = \int_{T_s - \bar{\tau}_{is}[t_k]T}^{T_s} e^{A_c\zeta} d\zeta B_c$.

Let us consider continuous-time dynamics (2.2). Using inter-sampling period T, the corresponding difference equation is given as:

$$x[t_k + T] = \tilde{A}x[t_k] + \tilde{B}u[t_k] + \tilde{E}\omega[t_k], \tag{2.17}$$

where the state and the input matrices are determined according to (2.4). Propagating (2.17) during the whole sampling period, one obtains

$$x[t_{k+1}] = \tilde{A}^N x[t_k] + \sum_{i=0}^{N-1} \tilde{A}^i \tilde{B} u[t_k] + \sum_{i=0}^{N-1} \tilde{A}^i \tilde{E} \omega[t_k],$$

under the assumption that $\omega[t_k + iT] = const.$, $\forall i \in \mathbb{Z}_{[0,N-1]}$. Regarding (2.3), it is obvious that $A = \tilde{A}^N$, $B = \sum_{i=0}^{N-1} \tilde{A}^i \tilde{B}$ and $E = \sum_{i=0}^{N-1} \tilde{A}^i \tilde{E}$.

In order to consider more general disturbance excitation, we can use the inter-sampling architecture in order to take into account the process disturbance variation between two consecutive samplings. Such a discrete-time model is obtain by propagating (2.17) during the whole sampling period and considering varying process noise between two consecutive samplings:

$$x[t_{k+1}] = Ax[t_k] + Bu[t_k] + \sum_{i=0}^{N-1} \tilde{A}^i \tilde{E} \omega[t_k + (N-1-i)T]. \tag{2.18}$$

The exogenous disturbance signal in the previous model approximates more acurately the continuous process noise $\omega(t)$. Moreover, the accuracy of the approximation is determined by the number of inter-sampling instants, N, which is a design parameter. A more detailed discussion on discrete-time representation for additive disturbance term will be provided in Chapter 3, where we consider set-theoretic methods.

Remark 2.5. In Lin and Antsaklis [2005] authors provided a similar discrete-time representation to (2.16), where they assumed that the control input is uniformly distributed over the whole interval between two consecutive samplings. However, according to the previous discussion, we can notice that the continuous-time dynamics (2.2) is approximated more accurately by (2.18).

Let us compare (2.18) with the following regular sampled-data system with respect to the sampling period T:

$$x[t_{k+1}] = Ax[t_k] + \sum_{i=0}^{N-1} \tilde{A}^i \tilde{B} u[t_k + (N-1-i)T] + \sum_{i=0}^{N-1} \tilde{A}^i \tilde{E} \omega[t_k + (N-1-i)T]. \tag{2.19}$$

Notice that the control signal is updated every T instant. Consequently, the network is more loaded, resulting in larger induced delays (for contention control networks for instance).

2.3 Stability analysis of NCS

In this section we provide some existing results for the stability analysis of NCS. The outlined results involve solely the discrete-time dynamics.

2.3.1 Second Lyapunov method

Let consider the following discrete-time dynamics:

$$x[t_{k+1}] = f(x[t_k], \omega[t_k]), \tag{2.20}$$

where $x[t_k] \in \mathbb{R}^n$ and $\omega[t_k] \in \mathbb{R}^p$ are the system state and the exogenous input at $t = t_k$, respectively. Variables x and ω represent sequences such that $\omega \in \mathcal{W}$, with $\mathcal{W} \subset \mathbb{R}^p$. Difference equation (2.20) can be regarded as a result of the closed-loop with a finite dimensional state-feedback controller.

Using the second Lyapunov method, stability of (2.20) can be verified without knowing its exact solution. Namely, if there exists a function with particular properties, such that it is decreasing along the system trajectory, then one can affirm stability of that system. This function is denoted as *Lyapunov function* (LF).

Disturbance attenuation analysis for the systems affected by bounded process noise requires some set-theoretic notions such as *robust invariant sets*.

Definition 2.1. *A set $\mathcal{X} \subseteq \mathbb{R}^n$ with $0 \in int(\mathcal{X})$ is called robustly positively invariant (RPI) set with respect to $\mathcal{W} \subseteq \mathbb{R}^p$ for system (2.20) if for all $x \in \mathcal{X}$, it holds that $f(x, \omega) \in \mathcal{X} \ \ \forall \omega \in \mathcal{W}$. If $\omega[t_k] = 0 \ \ \forall k \in \mathbb{Z}_+$ and $x \in \mathcal{X}$ implies $f(x) \in \mathcal{X}$, then $\mathcal{X}$ is referred to as positively invariant for system (2.20).*

The positive invariant notions are introduced here for the sake of *input-to-state* stability condition which is reported in the sequel. We consider positive invariance for linear dynamical systems in more details in Chapter 3.

Definition 2.2. *Let $\mathcal{X} \subseteq \mathbb{R}^n$ such that $0 \in int(\mathcal{X})$. The system (2.20) with $\omega[t_k] = 0 \ \ \forall k \in \mathbb{Z}_+$ is said to be asymptotically stable in $\mathcal{X}$ if there exists a $\mathcal{KL}$ function β such that, for each $x[t_0] \in \mathcal{X}$, it holds that $\|x[t_k]\| \leq \beta(\|x[t_0]\|, t_k) \ \ \forall k \in \mathbb{Z}_+$. If $\mathcal{X} = \mathbb{R}^n$ the system (2.20) is globally asymptotically stable.*

Definition 2.3. *Let $\bar{\omega}_{[t_k]} = \{\omega[t_i] \ : \ i \in \mathbb{Z}_{[0,k]}\}$, $\mathcal{X} \subseteq \mathbb{R}^n$ and $\mathcal{W} \subseteq \mathbb{R}^p$ such that $0 \in int(\mathcal{X})$. The system (2.20) is said to be input-to-state stable in $\mathcal{X}$ for inputs in $\mathcal{W}$ if there exists a $\mathcal{KL}$ function β and a $\mathcal{K}$ function γ such that, for each $x[t_0] \in \mathcal{X}$ and all $\omega[t_k] \in \mathcal{W}$, $k \in \mathbb{Z}_+$, it holds that $\|x[t_k]\| \leq \beta(\|x[t_0]\|, t_k) + \gamma(\|\bar{\omega}_{[t_{k-1}]}\|) \ \ \forall k \in \mathbb{Z}_+$. If $\mathcal{X} = \mathbb{R}^n$ and $\mathcal{W} = \mathbb{R}^p$, the system (2.20) is globally input-to-state stable.*

The following sufficient input-to-state stability condition holds (Jiang and Wang [2001], Lazar et al. [2009]).

Theorem 2.1. *Let $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$, $\gamma \in \mathcal{K}$ and $\mathcal{W} \subseteq \mathbb{R}^p$. Let $\mathcal{X}$ is a RPI set with respect to $\mathcal{W}$ such that $0 \in int(\mathcal{X})$. Let $V : \mathcal{X} \to \mathbb{R}_+$ be a function with $V(0) = 0$. Consider the following inequalities:*

$$\begin{aligned} \alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \\ V(f(x,\omega)) - V(x) \leq -\alpha_3(\|x\|) + \gamma(\|\omega\|). \end{aligned} \tag{2.21}$$

If the inequalities (2.21) hold $\forall x \in \mathcal{X}$ and $\forall \omega \in \mathcal{W}$, then the system (2.20) is input-to-state stable. If $\mathcal{X} = \mathbb{R}^n$ and $\mathcal{W} = \mathbb{R}^p$, then the system is globally input-to-state stable.

The function V that satisfies the Theorem 2.1 is called an *input-to-state Lyapunov function.*

2.3.2 Unperturbed NCS with constant delays

In this section we consider the stability analysis of NCS, with linear dynamics, under the assumption that the exogenous perturbation signal is zero. This assumption will be relaxed after we introduce necessary set-theoretic notions in Chapter 3.

Let us consider (2.9) such that $\tau[t_{k-j}] \in \mathbb{R}_{[0,\tau_{max}]}$, $\tau[t_{k-j}] = const.$, $\forall j \in \mathbb{Z}_{[0,d_m]}$. Assume that $\omega[t_k] = 0$, $\forall k \in \mathbb{Z}_+$. Then, (2.9) can be reformulated as

$$\xi[t_{k+1}] = \mathcal{A}\xi[t_k] + \mathcal{B}u[t_k], \tag{2.22}$$

where $\xi[t_k] = \begin{bmatrix} x^T[t_k] & u^T[t_{k-1}] & \dots & u^T[t_{k-d_m}] \end{bmatrix}^T$ and

$$\mathcal{A} = \begin{bmatrix} A & B_1 & \dots & B_{d_m-1} & B_{d_m} \\ 0_{m\times n} & 0_{m\times m} & \dots & 0_{m\times m} & 0_{m\times m} \\ 0_{m\times n} & I_m & \dots & 0_{m\times m} & 0_{m\times m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_{m\times n} & 0_{m\times m} & \dots & I_m & 0_{m\times m} \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} B_0 \\ I_m \\ 0_{m\times m} \\ \vdots \\ 0_{m\times m} \end{bmatrix}.$$

For the linear state feedback control $u = -Kx$, (2.22) can be reformulated as an autonomuos LTI system:

$$\xi_x[t_{k+1}] = \Lambda \xi_x[t_k], \tag{2.23}$$

where $\xi_x[t_k] = \begin{bmatrix} x^T[t_k] & x^T[t_{k-1}] & \dots & x^T[t_{k-d_m}] \end{bmatrix}^T$.

Analyzing stability of the augmented NCS model with constant delays, (2.23), can be preformed in several ways. The simplest approach consists in studying the eigenvalues of the transition matrix Λ (Schur method). This classical result is outlined in the following theorem (see e.g. Aström and Wittenmark [1997]).

Theorem 2.2. *System* (2.23) *is Schur-stable (globally asymptotically stable) if and only if* $\rho(\Lambda) < 1$.

Using Lyapunov stability method for the linear discrete-time systems, stability (2.23) can be also verified by solving a Linear Matrix Inequality (LMI).

Theorem 2.3. *System* (2.23) *is:*

- *stable (globally) if and only if there exists a positive definite matrix P such that*

$$\Lambda^T P \Lambda - P \preceq 0.$$

- *Schur-stable (globally asymptotically stable) if and only if there exists a positive definite matrix P such that*

$$\Lambda^T P \Lambda - P \prec 0.$$

The outlined stability analysis also applies to the models provided by Proposition 2.2, Proposition 2.3 and Proposition 2.4 if the network-induced delays are constant.

2.3.3 Unperturbed NCS with time-varying delays

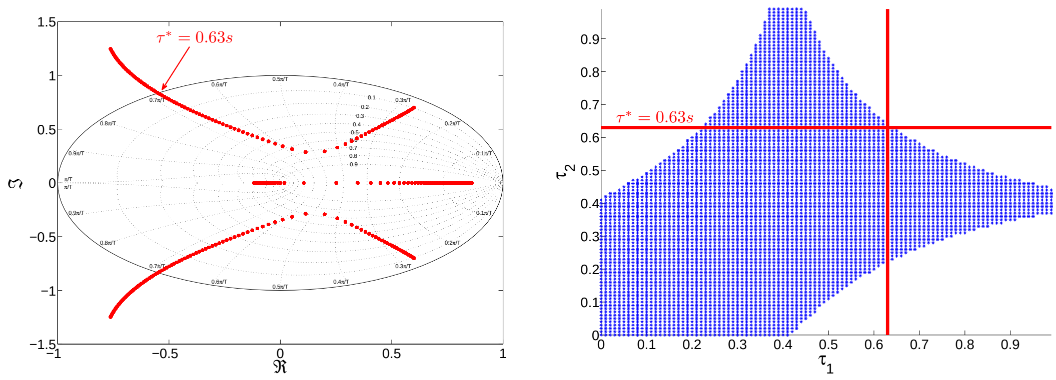
In most NCS realizations, network-induced delays are time-varying. Delay variation by itself can deteriorate system performance, even when it takes place among constant delays which independently do not cause instability of the closed-loop. This is shown by the following example.

Example 2.1. Consider the following sampled-data system:

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ -3.4 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ u[t_k] &= -\begin{bmatrix} -2.8962 & -0.3785 \end{bmatrix} x[t_k],\end{aligned}$$

with the sampling period $T_s = 1\ s$.

If the closed-loop dynamics is affected by a constant delay, one can evaluate a delay margin, $\tau^* \in \mathbb{R}_{(0,\tau_{max}]}$, i.e., the maximal allowable constant delay such that the closed-loop system is stable (see Fig. 2.7a). However, the delay margin is not relevant anymore for the stability analysis when the induced delays are varying, where even much smaller delays than τ^* can degrade performance of the closed-loop system. Namely, let assume



(A) Root locus for $\tau \in \mathbb{R}_{[0,T_s]}$

(B) Stability region (in blue) with respect to delay variation $\tau_1 \to \tau_2$

FIGURE 2.7: Example 2.1

that $\tau[t_k] = \tau_1 \in \mathbb{R}_{(0,T_s)}$ and $\tau[t_{k+1}] = \tau_2 \in \mathbb{R}_{(0,T_s)}$. The discrete-time closed-loop realization for the two consecutive steps is:

$$\xi_x[t_{k+2}] = \Lambda(\tau_2)\Lambda(\tau_1)\xi_x[t_k],$$

where $\xi_x = \begin{bmatrix} x^T[t_k] & x^T[t_{k-1}] \end{bmatrix}^T$ and $\Lambda(\tau_1)$ and $\Lambda(\tau_2)$ are the transition matrices determined according to:

$$\Lambda(\tau) = \begin{bmatrix} A - B_0(\tau)K & -B_1(\tau)K \\ I_n & 0_{n\times n} \end{bmatrix}$$

For $\tau_1 = 0.5\ s$ and $\tau_2 = 0.1\ s$, $\Lambda(\tau_1)$ and $\Lambda(\tau_2)$ are Schur-stable, while $\Lambda(\tau_2)\Lambda(\tau_1)$ is not. This is explained by the fact that, in general case, multiplication of two Schur-stable matrices is not a Schur-stable matrix (see Liberzon [2003]). Stability region in delay-parameter space with respect to τ_1 and τ_2 and for two consecutive sampling instants is shown in Fig. 2.7b. Notice that stability can be even more compromised for more sampling instants and different combinations of the two matrices.

Let us consider the general NCS model (2.9) with time-varying delay $\tau[t_{k+j-d_m}] \in \mathbb{R}_{[0,\tau_{max}]}$, $\forall j \in \mathbb{Z}_{[0,d_m]}$ and $\omega[t_k] = 0$, $\forall k \in \mathbb{Z}_+$. Equivalent representation in the augmented state-space is given as:

$$\xi[t_{k+1}] = \mathcal{A}(\tau_k)\xi[t_k] + \mathcal{B}(\tau_k)u[t_k], \tag{2.24}$$

where $\xi[t_k] = \begin{bmatrix} x^T[t_k] & u^T[t_{k-1}] & \dots & u^T[t_{k-d_m}] \end{bmatrix}^T$ and

$$\mathcal{A}(\tau_k) = \begin{bmatrix} A & B_1(\tau_k) & \dots & B_{d_m-1}(\tau_k) & B_{d_m}(\tau_k) \\ 0_{m\times n} & 0_{m\times m} & \dots & 0_{m\times m} & 0_{m\times m} \\ 0_{m\times n} & I_m & \dots & 0_{m\times m} & 0_{m\times m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_{m\times n} & 0_{m\times m} & \dots & I_m & 0_{m\times m} \end{bmatrix}, \quad \mathcal{B}(\tau_k) = \begin{bmatrix} B_0(\tau_k) \\ I_m \\ 0_{m\times m} \\ \vdots \\ 0_{m\times m} \end{bmatrix}.$$

For the linear state-feedback control $u = -Kx$, (2.24) can be reformulated as:

$$\xi_x[t_{k+1}] = \Lambda(\tau_k)\xi_x[t_k], \tag{2.25}$$

where $\xi_x[t_k] = \begin{bmatrix} x^T[t_k] & x^T[t_{k-1}] & \dots & x^T[t_{k-d_m}] \end{bmatrix}^T$.

To guarantee the stability of (2.25), it is sufficient to prove that this system admits a common quadratic Lyapunov function, $V(\xi_x[t_k]) = \xi_x^T[t_k]P\xi_x[t_k]$ such that

$$\Lambda(\tau_k)^T P \Lambda(\tau_k) - P \prec 0, \quad \forall \tau_k \in \mathbb{R}^{d_m+1}_{[0,\tau_{max}]}. \tag{2.26}$$

Since the delay parameter is defined on an interval, the transition matrix $\Lambda(\tau_k)$ can have an infinite number of realizations. Consequently, condition (2.26) involves an infinite number of LMIs. The same holds for (2.14) with $\tau_{is} \in \mathbb{R}_{[0,T_s)}$. By using convex-embedding approach, one can reduce these conditions down to a finite number. The idea is to embed $\Lambda(\tau_k)$ in the following polytope:

$$\bar{\Lambda} = \left\{ \sum_{i=1}^{\nu} q_i \Lambda_i \; : \; \Lambda_i \in \mathbb{R}^{n(d_m+1)\times n(d_m+1)}, \quad \sum_{i=1}^{\nu} q_i = 1 \right\}$$

such that

$$\Lambda(\tau_k) \in \bar{\Lambda}, \quad \forall \tau_k \in (\mathbb{R}_{[0,\tau_{max}]})^{d_m+1}.$$

Matrices Λ_i are the generators (vertices) for the convex set $\bar{\Lambda}$. Consequently, the stability of (2.25) is determined by a finite number of these generators.

On the other side, discrete-time representations (2.12) and (2.16) are *switched* dynamics, i.e., they are already determined by a finite number of modes because $\bar{\tau}_k$ is defined on a finite grid of delay parameter space $\mathbb{R}^{d_m+1}_{[0,\tau_{max}]}$ (determined by N) and $\bar{\tau}_{is} \in \mathbb{Z}_{[0,N-1]}$, respectively. Moreover, dynamics (2.12) and (2.16) do not involve any over-approximations but represent the exact closed-loop dynamics.

There have been proposed different convex over-approximation methods in the literature so far. For instance, over-approximation based on Jordan decomposition (see Olaru and Niculescu [2008], Cloosterman et al. [2009], Van de Wouw et al. [2010] and Lombardi [2011]), over-approximation based on Taylor series expansion (see Hetel et al. [2006]), over-approximation based on Cayley-Hamilton theorem (see Gielen et al. [2010]) etc. For an overview and comparison of available convex over-approximation methods, we refer to Heemels et al. [2010].

Stability analysis can be verified according to the following theorem (see e.g. Liberzon [2003]).

Theorem 2.4. *Given a finite set of square matrices* $\{\Lambda_1, \Lambda_2, \dots, \Lambda_\nu\}$*, the corresponding discrete-time representation is stable if* Λ_i*,* $\forall i \in \mathbb{Z}_{[1,\nu]}$ *share a common Lyapunov function, equivalently, if there exists a positive definite* P *such that*

$$\Lambda_i^T P \Lambda_i - P \prec 0, \quad \forall i \in \mathbb{Z}_{[1,\nu]}. \tag{2.27}$$

The class of quadratic Lyapunov function is sufficient to prove stability of (2.25). However, by using a different class of Lyapunov functions, it is possible to obtain necessary and sufficient condition (see Blanchini [1995]). Further details on this subject will be presented in the subsequent chapter where the set-theoretic notions are considered.

Apart from Lyapunov theory, one can consider another, non-conservative, stability analysis approach for a NCS determined by a finite set of transition matrices. This approach is based on the *joint spectral radius* of all possible products of generating matrices. Let $\mathcal{I}_k = \{(i_1, i_2, \dots, i_k) \ : \ i_j \in \mathbb{Z}_{[1,\nu]}\}$ represent the set of all possible choices of k indices from the set $\mathbb{Z}_{[1,\nu]}$. Denote by $C_k = (i_1, i_2, \dots, i_k) \in \mathcal{I}_k$ one such a choice which admits the following matrix product:

$$\Pi_{C_k} = \Lambda_{i_1} \Lambda_{i_2} \dots \Lambda_{i_k}.$$

The following notion was introduced in Rota and Strang [1960].

Definition 2.4. *Given a finite set of square matrices* $\{\Lambda_1, \Lambda_2, \dots, \Lambda_\nu\}$*, the quantity*

$$\rho(\Lambda_1, \Lambda_2, \dots, \Lambda_\nu) = \limsup_{k \geq 0} \max_{C_k \in \mathcal{I}_k} \rho(\Pi_{C_k})^{\frac{1}{k}}$$

is said to be the joint spectral radius of that set.

Theorem 2.5. *A switched linear system* $\xi[t_{k+1}] = \Lambda_{\sigma(k)} \xi[t_k]$*, where* $\Lambda_{\sigma(k)} \in \{\Lambda_1, \Lambda_2, \dots, \Lambda_\nu\}$ *is globally asymptotically stable under arbitrary switching if and only if there exists a finite* s *such that*

$$\rho\left(\Lambda_{i_1} \Lambda_{i_2} \dots \Lambda_{i_s}\right) < 1, \quad \forall \Lambda_{i_j} \in \{\Lambda_1, \Lambda_2 \dots, \Lambda_\nu\}$$

for $j = 1, \dots, s$*.*

It is worth pointing out that Theorem 2.5, although provides necessary and sufficient condition, is not easy to verify numerically since it is an NP-hard problem (see Blondel and Tsitsiklis [1997]) to compute or to approximate. In addition, the result does not give any clue on how large the integer s needs to be.

In the following example, which was proposed by Olaru and Niculescu [2008], we consider the difference in stability analysis between models obtained by the convex embedding and the inter-sampling method ((2.9), (2.14) and (2.12), (2.16), respectively).

Example 2.2. Consider the following continuous-time plant

$$\dot{x}(t) = A_c x(t) + B_c u(t - \tau)$$

where $A_c = \begin{bmatrix} 1.1 & -0.1 \\ 1 & 0 \end{bmatrix}$, $B_c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $T_s = 0.1$ s and $\tau \in \mathbb{R}_{[0,T_s]}$.

Polytopic over-approximation is determined for the following augmented state-space representation of the system:

$$\begin{bmatrix} x[t_{k+1}] \\ u[t_k] \end{bmatrix} = \begin{bmatrix} e^{A_c T_s} & B - \Delta(\tau) \\ 0_{1\times 2} & 0 \end{bmatrix} \begin{bmatrix} x[t_k] \\ u[t_{k-1}] \end{bmatrix} + \begin{bmatrix} \Delta(\tau) \\ 1 \end{bmatrix} u[t_k], \tag{2.28}$$

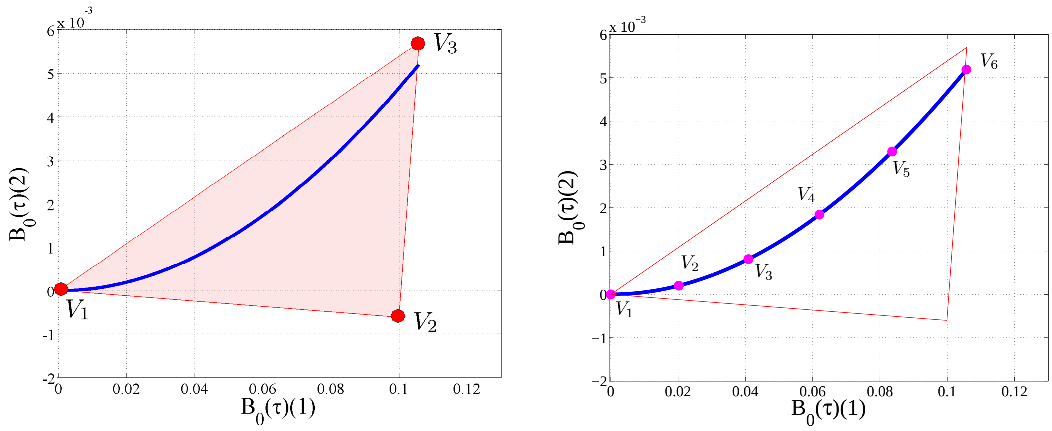
where

$$\Delta(\tau) = \int_0^{T_s - \tau[t_k]} e^{A_c \zeta} d\zeta B_c, \quad B = \int_0^{T_s} e^{A_c \zeta} d\zeta B_c.$$

Inter-sampling model is obtained as:

$$\begin{bmatrix} x[t_{k+1}] \\ u[t_k] \end{bmatrix} = \begin{bmatrix} e^{A_c T_s} & B_1(\bar{\tau}_{is}) \\ 0_{1\times 2} & 0 \end{bmatrix} \begin{bmatrix} x[t_k] \\ u[t_{k-1}] \end{bmatrix} + \begin{bmatrix} B_0(\bar{\tau}_{is}) \\ 1 \end{bmatrix} u[t_k], \tag{2.29}$$

where $\bar{\tau}_{is} \in \mathbb{Z}_{[0,N-1]}$. Result of the polytopic over-approximation is shown in Fig. 2.8a, which is obtained according to the method proposed by Olaru and Niculescu [2008]. According to this approach, matrix $\Delta(\tau)$ is embedded into a simplex, thus providing the simplest possible result from numerical efficiency point of view. Obviously, in this example, the existence of a common Lyapunov function should be examined only with respect to three transition matrices with parameters on the vertices of the polytope depicted on Fig. 2.8a. Using inter-sampling approach, the same region of uncertainty is discretized (instead of the convex embedding) and shown on Fig. 2.8b. It is obvious that the convex embedding approach involves less different modes, but on the other hand, it is more conservative with respect to the considered region of uncertainty. We know that if two quadratic matrices share a quadratic common Lyapunov function, then any convex combination of these matrices is Schur. Therefore, if there exist common Lyapunov functions with respect to (2.28) and (2.29), they would guarantee stability also for the regions depicted on Fig. 2.8b. From this analysis, we can conclude that the inter-sampling approach provides less conservative results at the cost of more complex numerical calculation.



(A) Convex embedding

(B) Inter-sampling

FIGURE 2.8: Approximation of the uncertain delay parameter

2.4 Concluding remarks

It has been shown that the performance of the NCSs greatly depends on the underlaying network protocol, i.e., on network-induced delays (constant or time-varying), bandwidth limitation or packet dropouts. In this section we considered modeling for the NCSs, with linear dynamics, with respect to network-induced delays and packet dropouts. Obtained models are based on assumption that outdated measurements and control inputs are rejected. Even though the implementation of this approach would require non-standard network protocols, in the author's opinion, such an approach would be more suitable for the real-time control applications. Namely, by rejecting outdated information, network resources are used only for the transmission of up-to-date messages when this is permitted by the network. Thus, an additional network load is avoided.

We considered a delay parameter which is uncertain and bounded on an interval. This uncertainty is incorporated in a model, which, consequently, has an infinite number of possible modes. In order to obtain a suitable mathematical model for the stability analysis, two approaches have been proposed. The first approach exploits the over-approximation method. Namely, the delay-dependent parameter is bounded by a convex region. Therefore, the stability with respect to all possible delay variations is guaranteed by the stability of generators for that convex set. Quality of the approximation and numerical cost are conflicting requirements and they depend on the number of generators. The main advantage of this approach is that it takes into consideration all possible delays from the interval at a relatively low numerical cost. On the other side, it is more conservative with respect to the quality of approximation since it takes into consideration larger uncertainty region. The second approach, based on the inter-sampling strategy, does not require any over-approximation since the obtained models already have a finite number of different switching modes. However, number of these modes can be larger for an acceptable approximation quality than it is the case for the over-approximation method. The main advantage of this approach is that it corresponds to the system configuration, since the introduced buffers allow only certain delays from the the delay interval, in particular, those that are on the grid determined by the number of inter-samplings.

One of the goals of the thesis is to propose a FTC[6]-like scheme for delay detection and compensation in NCS. This approach, proposed for the single sensor (Chapter 5) and the multi-sensor (Chapter 6) architectures, greatly exploits the inter-sampling configuration. Hence the model introduced by Proposition 2.4 will be widely used throughout this study.

[6]Fault tolerant control

Chapter 3

Set-theoretic methods for NCS

POSITIVE invariance is a widely used concept in control and it provides an effective way to address constrained control analysis and design problems. For instance, hard physical constraints imposed to a dynamical system can be satisfied by ensuring the existence of a positively invariant set inside the admissible region. In this way one directly defines a set of initial conditions for which the dynamics satisfies constraints for all future instants (for an overview on set-theoretic methods in control we refer to Blanchini [1999]). This remarkable property of positively invariant sets found application in many other control areas such as model predictive control (see e.g. Kothare et al. [1996], Mayne et al. [2000], Blanchini and Miani [2008]), fault tolerant control (see e.g. Seron et al. [2008], Martınez et al. [2008], Olaru et al. [2010], Ocampo-Martinez et al. [2010], Gaspar et al. [2012]), reference governor design (see Stoican et al. [2012], Stanković et al. [2012]).

Concerning discrete-time systems with delays, there are two main ideas in the existing literature on how to address the problem of positive invariance. The first approach relies on rewriting such a system in the augmented state space (see Section 2.3) and to consider it as a regular linear system (with or without uncertainties). On the other hand, the second approach, also referred to as *D-invariance*, considers invariance directly in the initial state-space. A pioneering work on D-invariance was published by Dambrine et al. [1995a] and Dambrine et al. [1995b], where the authors considered stabilizing control design for constrained continuous-time systems with delays. These results were followed by Hennet and Tarbouriech [1998] for the continuous-time delay-difference equations. The indicated initial works reported an interest to extend some already existing constrained control notions to systems with delays. This interest is still actual today and it is reflected in several recent works by Gielen et al. [2012b], Lombardi et al. [2011b], Rakovic et al. [2012], Stanković et al. [2011]. However, characterization of D-invariant sets is still an open problem in control, even for linear discrete-time systems. As an example where D-invariant sets are useful, let us consider the following MPC design problem for a NCS model $x[t_{k+1}] = A[t_k] + \sum_{j=0}^{d_m} B_{d_m-j}u[t_{k+j-d_m}],\ x \in \mathcal{X}_x,\ u \in \mathcal{X}_u$ with constant delays and with state and input constraints (see e.g. Mayne et al. [2000]). The control sequence on a finite horizon is obtained by solving the following optimization problem:

$$\min_u \sum_{s=k}^{k+N-1} l\left(x[t_s], u[t_k], \ldots, u[t_{s-d_m}]\right) + T\left(x[t_{k+N}], u[t_{k+N}] \ldots, u[t_{k+N-d_m}]\right), \quad N > d_m \tag{3.1}$$

subject to

$$\begin{aligned} x[t_{s+1}] &= Ax[t_s] + \sum_{j=0}^{d_m} B_{d_m-j}u[t_{s+j-d_m}], \quad \forall s \in \mathbb{Z}_{[k,k+N-1]} \\ \begin{bmatrix} x[t_s]^T & \ldots & u[t_{s-d_m}]^T \end{bmatrix}^T &\in \mathcal{X}_x \times \ldots \times \mathcal{X}_u, \quad u[t_s] \in \mathcal{X}_u \\ \begin{bmatrix} x[t_{k+N}]^T & \ldots & u[t_{k+N-d_m}]^T \end{bmatrix}^T &\in \mathcal{X}, \end{aligned} \tag{3.2}$$

where $l(\cdot) > 0$, $T(\cdot) > 0$ are the stage and the terminal cost in the objective function, while $\mathcal{X}$ is a positively invariant terminal set. This set can be defined either in the augmented state-space as $\mathcal{X} \subseteq (\mathbb{R}^n)^{d_m+1}$ or as $\mathcal{X} = \mathcal{Y} \times \ldots \mathcal{Y} \subseteq (\mathbb{R}^n)^{d_m+1}$, where $\mathcal{Y} \subseteq \mathbb{R}^n$ is a D-invariant set. Both formulations fulfill the original objective: enforcing the containment of the trajectories in a terminal invariant structure in order to ensure the stability of the control law. However, they present different characteristics, the D-invariance being preferable from the complexity point of view, while the positively invariant set for the augmented representation describes generally a larger domain in the extended state space.

There are two main objectives of this chapter. One is to provide a support in terms of invariant[1] sets for fault and delay detection algorithms that will be be considered in the Chapter 5 and Chapter 6. Related to this subject, required results are already well-known in the literature and they include positive invariance for linear systems with additive disturbance. The second objective is to induce a general positive invariance notion for the NCS models presented in the previous chapter. For this purpose, we provide an overview of different concepts of invariance for discrete-time systems with delays[2]. Moreover, we detail some new insights on the existence and construction, in particular for $D-$invariant sets related to the linear delay-difference equations with additive disturbance. Furthermore, some alternative invariance solutions are discussed as well.

[1]In this work we are strictly interested in the concept of "positive invariance" which refers to the forward evolution of a system. However, notation "invariance" is sometimes used for brevity.

[2]The class of discrete-time systems with delays is considered as the general class of discrete-time NCS models

3.1 Basic set theoretic notions

The following standard definitions from the theory of sets are introduced. For further details on the set-related notions we refer to Kolmogorov and Fomin [1999], Boyd and Vandenberghe [2004].

Definition 3.1. *A set $\mathcal{X} \subset \mathbb{R}^n$ is bounded if $\|x_1 - x_2\|_p < \infty$, $\forall x_1, x_2 \in \mathcal{X}$.*

Definition 3.2. *A set $\mathcal{X} \subset \mathbb{R}^n$ is closed if $\forall x \notin \mathcal{X}$ $\exists \varepsilon \in \mathbb{R}^+$ such that $\mathbb{B}^p_\varepsilon(x) \notin \mathcal{X}$.*

Definition 3.3. *A set $\mathcal{X} \subset \mathbb{R}^n$ is compact if is bounded and closed.*

Definition 3.4. *A set $\mathcal{X} \subset \mathbb{R}^n$ is convex if $\forall x_1, x_2 \in \mathcal{X}$ the convex combination satisfies*

$$\theta x_1 + (1-\theta)x_2 \in \mathcal{X}, \quad \forall \theta \in \mathbb{R}_{[0,1]}.$$

Definition 3.5. *A set $\mathcal{X} \subset \mathbb{R}^n$ is a (proper) C-set if is convex, compact and includes the origin in its strict interior.*

Definition 3.6. *The convex hull of the set $\mathcal{X} \subset \mathbb{R}^n$ is defined as:*

$$ConvHull(\mathcal{X}) = \left\{ \theta_1 x_1 + \ldots + \theta_k x_k \;:\; x_i \in \mathcal{X},\; \theta_i \geq 0,\; \sum_{i=1}^{k} \theta_i = 1 \right\}.$$

Definition 3.7. *The support function of a set $\mathcal{X} \subset \mathbb{R}^n$ at $z \in \mathbb{R}^n$ is defined as:*

$$\phi_{\mathcal{X}}(z) = \sup_{x \in \mathcal{X}} \; z^T x.$$

Regarding the class of convex sets, basic set-operations are defined by the following definitions (see e.g. Schneider [1993]).

Definition 3.8. *Given two convex set $\mathcal{X} \subset \mathbb{R}^n$ and $\mathcal{Y} \subset \mathbb{R}^n$, their intersection is defined as*

$$\mathcal{X} \cap \mathcal{Y} = \{x \;:\; x \in \mathcal{X},\; x \in \mathcal{Y}\}.$$

Definition 3.9. *Given two convex sets $\mathcal{X} \subset \mathbb{R}^n$ and $\mathcal{Y} \subset \mathbb{R}^n$, the Minkowski (geometric) sum is defined as*

$$\mathcal{X} \oplus \mathcal{Y} = \{z = x + y \;:\; \forall x \in \mathcal{X},\; \forall y \in \mathcal{Y}\}.$$

Definition 3.10. *Given two convex sets $\mathcal{X} \subset \mathbb{R}^n$ and $\mathcal{Y} \subset \mathbb{R}^n$, the Pontryagin difference of $\mathcal{X}$ and $\mathcal{Y}$ is defined as*

$$\mathcal{X} \ominus \mathcal{Y} = \{x \;:\; x + y \in \mathcal{X},\; \forall y \in \mathcal{Y}\}.$$

Definition 3.11. *Given a convex set $\mathcal{X} \subset \mathbb{R}^n$ and an affine map $M : \mathbb{R}^n \to \mathbb{R}^m$, the set image under the map M is defined as*

$$M(\mathcal{X}) = \{y \;:\; y = M(x),\; \forall x \in \mathcal{X}\}.$$

The following properties of the Minkowski sum will be used throughout this study.

Lemma 3.1. *Let $\mathcal{X} \subset \mathbb{R}^n, \mathcal{Y} \subset \mathbb{R}^n$ and $\mathcal{Z} \subset \mathbb{R}^n$ are $C-$sets, $\alpha \in \mathbb{R}_{(0,\infty)}$, $\beta \in \mathbb{R}_{(0,\infty)}$ such that $\alpha \geq \beta$ and $M \in \mathbb{R}^{n\times n}$, $N \in \mathbb{R}^{n\times n}$. Then, $\mathcal{X}\oplus\mathcal{Y} = \mathcal{Y}\oplus\mathcal{X}$, $(\mathcal{X}\oplus\mathcal{Y})\oplus\mathcal{Z} = \mathcal{X}\oplus(\mathcal{Y}\oplus\mathcal{Z})$, $\alpha\mathcal{X} \oplus \beta\mathcal{X} = (\alpha+\beta)\mathcal{X}$, $M(\mathcal{X} \oplus \mathcal{Y}) = M\mathcal{X} \oplus M\mathcal{Y}$. Furthermore, $\mathcal{X} \oplus \mathcal{Z} \subseteq \mathcal{Y} \oplus \mathcal{Z}$ if and only if $\mathcal{X} \subseteq \mathcal{Y}$ and $M\mathcal{X} \oplus N\mathcal{X} \subseteq M\mathcal{Y} \oplus N\mathcal{Y}$ if and only if $\mathcal{X} \subseteq \mathcal{Y}$.*

The most exploited classes of convex sets in control are ellipsoids and polyhedral sets. Ellipsoids are preferred due to their simpler representation. They are also associated with quadratic Lyapunov functions and LMIs, which allows an efficient numerical manipulation. However, when compared with polyhedral sets, ellipsoids may admit more conservative representations, for instance when an approximation of a convex region is required. Furthermore, ellipsoids do not form a closed family with most of the elementary set operators such as Minkowski sum, Pontryagin difference or affine set image. On the other side, implementation of these operators is simpler for polyhedral sets (see Kvasnica et al. [2004]). Their dual representation (see Ziegler [1995]) also allows choosing the most suitable form for a particular problem. The main disadvantage of polyhedral sets comes from possible complexity of their representations but it can be handled via specific subclasses as for instance *zonotopes*.

In this work we utilized polyhedral sets, even though most of the presented results (in particular those outlined in Chapter 5 and Chapter 6) are independent on the utilized set representation.

Definition 3.12. *A hyperplane $H \subset \mathbb{R}^n$ is a set of the form*

$$H = \{x \in \mathbb{R}^n \;:\; f_i^T x = g_i\},$$

where $f_i \in \mathbb{R}^n$ is a column vector.

Definition 3.13. *A closed half-space $H \subset \mathbb{R}^n$ is a set of the form*

$$H = \{x \in \mathbb{R}^n \;:\; f_i^T x \leq g_i\},$$

where $f_i \in \mathbb{R}^n$ is a column vector.

Definition 3.14. *A convex set $\mathcal{P}(F,g) \subset \mathbb{R}^n$ is a set of the form*

$$\mathcal{P}(F,g) = \{x \in \mathbb{R}^n \;:\; f_i^T x \leq g_i,\; i \in \mathbb{Z}_{[1,q]}\},$$

where $f_i \in \mathbb{R}^n$ is the i^{th} column of $F^T \in \mathbb{R}^{n\times q}$ and g_i is the i^{th} element of $g \in \mathbb{R}^q$.

Definition 3.14 specifies a half-space representation of a polyhedral set. Each polyhedral set can also be defined by the vertex representation.

Definition 3.15. *For a given $V \in \mathbb{R}^{n\times s}$, a convex set $\mathcal{P}(V) \subset \mathbb{R}^n$ is defined as a set of the form*

$$\mathcal{P}(V) = \{x \in \mathbb{R}^n \ : \ x = \sum_{i=1}^{s} \theta_i v_i, \ v_i \in \mathbb{R}^n, \ 0 \leq \theta_i \leq 1, \ \sum_{i=1}^{s} \theta_i = 1\},$$

where $v_i \in \mathbb{R}^n$ represents the i^{th} column of V.

Definition 3.16. *A polytope is a bounded polyhedral set.*

Proposition 3.1. *A polyhedral set $\mathcal{P}(V)$ is a polytope.*

Dual representation of polyhedral sets is an important feature that provides flexible and efficient implementation of basic set-operations. Thus, for instance, some set-operations can be more efficiently implemented by using one representation while, at the same time, their realization can be cumbersome by using the other one. We define in the sequel (by using the most appropriate set representation) some set operators that are used throughout this study.

Let M denote a linear map and $\alpha \in \mathbb{R}_{[0,\infty]}$. As we highlighted before, polyhedral sets form a closed family for the following set operations.

- For $\mathcal{X} = \mathcal{P}(V)$, the image $M(\mathcal{X})$ is a polyhedral set determined by

$$M(\mathcal{X}) = \mathcal{P}(MV);$$

- For $\mathcal{X} = \mathcal{P}(F, g)$, the pre-image $M^{-1}(\mathcal{X})$ is a polyhedral set determined by

$$M^{-1}(\mathcal{X}) = \mathcal{P}(FM, g);$$

- For $\mathcal{X} = \mathcal{P}(F_x, g_x)$ and $\mathcal{Y} = \mathcal{P}(F_y, g_y)$, the intersection $\mathcal{X} \cap \mathcal{Y}$ is a polyhedral set determined by

$$\mathcal{X} \cap \mathcal{Y} = P\left(\begin{bmatrix} F_x \\ F_y \end{bmatrix}, \begin{bmatrix} g_x \\ g_y \end{bmatrix}\right);$$

- For $\mathcal{X} = \mathcal{P}(V)$, the scaled set $\alpha\mathcal{X}$ is a polyhedral set determined by

$$\alpha\mathcal{X} = \mathcal{P}(\alpha V), \ \forall \lambda \in \mathbb{R}_{[0,\infty)} \ \text{and} \ \alpha\mathcal{X} = \mathcal{P}\left(\frac{F}{\alpha}, g\right), \ \forall \alpha \in \mathbb{R}_{(0,\infty)};$$

- For $\mathcal{X} = \mathcal{P}(V_x)$ and $\mathcal{Y} = \mathcal{P}(V_y)$, the Minkowski sum $\mathcal{X} \oplus \mathcal{Y}$ is a polyhedral set determined by

$$\mathcal{X} \oplus \mathcal{Y} = \sum_i \theta_i v_{xi} + \sum_j \vartheta_j v_{yj} = \sum_i \sum_j \vartheta_j \theta_i v_{xi} + \sum_i \sum_j \theta_i \vartheta_j v_{yj}$$
$$= \sum_i \sum_j \theta_i \vartheta_j (v_{xi} + v_{yj}),$$

where $\sum_i \sum_j \theta_i \vartheta_j = 1$, $\theta_i \vartheta_j \in \mathbb{R}_{[0,1]}$ since $\sum_i \theta_i = 1$, $\theta_i \in \mathbb{R}_{[0,1]}$ and $\sum_j \vartheta_j = 1$, $\vartheta_j \in \mathbb{R}_{[0,1]}$.

- For $\mathcal{X} = \mathcal{P}(F_x, g_x)$ and $\mathcal{Y} = \mathcal{P}(F_y, g_y)$, the Pontryagin difference $\mathcal{X} \ominus \mathcal{Y}$ is a polyhedral set determined by

$$\mathcal{X} \ominus \mathcal{Y} = \mathcal{P}(F_x, \bar{g}),$$

where $\bar{g} = g_x - \max\limits_{y \in \mathcal{P}(F_y, g_y)} F_x y$;

- For $\mathcal{X} = \mathcal{P}(V_x)$ and $\mathcal{Y} = \mathcal{P}(V_y)$, the convex hull $ConvHull\,(\mathcal{X} \cup \mathcal{Y})$ is a polyhedral set determined by
$$ConvHull\,(\mathcal{X} \cup \mathcal{Y}) = \mathcal{P}\left(\begin{bmatrix} V_x & V_y \end{bmatrix}\right).$$

In order to verify the inclusion between polyhedral sets, we employ the following proposition which is reported without proof (see Kolmanovsky and Gilbert [1998]).

Proposition 3.2. *Let $\mathcal{X} = \mathcal{P}(F_x, g_x) = \{x \ : \ F_{xi} x \leq g_{xi},\ i \in \mathbb{Z}_{[1,q]}\}$. Then $\mathcal{Y} \subseteq \mathcal{X}$ if and only if*
$$\phi_{\mathcal{Y}}(f_{xi}) \leq g_{xi}, \quad \forall i \in \mathbb{Z}_{[1,q]}.$$

The main drawback of the polyhedral sets is their increased complexity in representation. Hence, it is important to provide as simple as possible representation of a polyhedral set before running an algorithm. Such a representation, which is unique, is denoted as the *minimal representation* of a polyhedral set and it is stated by the following definition.

Definition 3.17. *A half-space or vertex representation is minimal if there is no other representation of the same set involving $F \in \mathbb{R}^{q_1 \times n}$ or $V \in \mathbb{R}^{n \times s_1}$ such that $q_1 < q$ or $s_1 < s$, respectively.*

In general, the algorithms which involve polytopes are very demanding in terms of computational complexity. Therefore, it is often mandatory to work with the minimal representation to keep the complexity as low as possible Blanchini and Miani [2008].

For a polytope which is determined by intersections of half-spaces, the minimal representation is reported in a more specific way through a *normalized representation.*

Definition 3.18. *A polytope $\mathcal{X}(F,g)$ is in a normalized representation if it has the following property:*

$$F_i F_i^T = 1,\ \forall i \in \mathbb{Z}_{[1,q]}.$$

A normalized representation is a unique minimal representation of a polytope.

In spite of their complexity, it was already mentioned that the polyhedral sets are more flexible when compared to the ellipsoidal sets. This is noticeable from the fact that a compact and convex set can be arbitrarily closely approximated by a polyhedral set. Regarding the approximation, we will particularly use outer approximations of a polytope. Namely, if $\mathcal{Y}$ is a C-set, then $\forall \epsilon \in \mathbb{R}_{(0,1)}$ there exists a polytope $\mathcal{X}$ such that

$$\mathcal{Y} \subseteq \mathcal{X} \subseteq (1+\epsilon)\mathcal{Y},$$

where $\mathcal{X}$ is an outer approximation of $\mathcal{Y}$.

3.2 Positive invariance for LTI systems. Prerequisites and preliminaries

In this section we address the problem of the existence and the construction of positively invariant sets for LTI systems with additive disturbance. This is a well-known problem in the control theory, therefore most of the results are reported without entering too much into details. Notions considered in this section will be used in Chapter 5 and Chapter 6 for designing a fault and delay tolerant control algorithm for NCSs with network-induced sensor-to-controller delay.

Consider the following discrete-time LTI system with additive disturbance:

$$x[t_{k+1}] = Ax[t_k] + Bu[t_k] + E\omega[t_k], \tag{3.3}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $E \in \mathbb{R}^{n\times p}$ and $\omega[t_k] \in \mathcal{W}$, with $\mathcal{W}$ as a bounded $C-$set. Notice that (3.3) corresponds to a NCS model with constant delay (see Section 2.3.2).

A stabilizing control signal for (3.3) is determined by $u[t_k] = u_{ref}[t_k] - K(x[t_k] - x_{ref}[t_k])$, where

$$x_{ref}[t_{k+1}] = Ax_{ref}[t_k] + Bu_{ref}, \tag{3.4}$$

defines the reference dynamics that needs to be followed as close as possible by (3.3). The control performances are determined by the tracking error $z = x - x_{ref}$ which yields the following dynamics:

$$z[t_{k+1}] = (A - BK)z[t_k] + E\omega[t_k]. \tag{3.5}$$

Assume that there exists a set $\mathcal{Z} \subset \mathbb{R}^n$ such that $z[t_{k+1}] \in \mathcal{Z},\ \forall z[t_k] \in \mathcal{Z}$ and $\forall \omega[t_k] \in \mathcal{W}$. Then, the tracking error dynamics stays within $\mathcal{Z}$ for any initial condition from that set, i.e., $\forall x[t_k] \in \{x_{ref}[t_k]\} \oplus \mathcal{Z}$

$$x[t_{k+1}] \in \{x_{ref}[t_{k+1}]\} \oplus \mathcal{Z}, \quad \forall \omega[t_k] \in \mathcal{W}. \tag{3.6}$$

For more details we refer to Mayne et al. [2005].

The set $\mathcal{Z}$ is referred to as *robust positively invariant set* for (3.5) and it is formally defined by the following definition.

Definition 3.19. *The $C-$set $\mathcal{Z} \subseteq \mathbb{R}^n$ is said to be robustly positively invariant (RPI) with respect to (3.5) if $\forall z[t_k] \in \mathcal{Z}$ and $\forall \omega[t_k] \in \mathcal{W}$, $z[t_{k+1}] \in \lambda\mathcal{Z}$, where $\lambda \in \mathbb{R}_{[0,1]}$, i.e., if $(A - BK)\mathcal{Z} \oplus \mathcal{W} \subseteq \lambda\mathcal{Z}$. For $\lambda \in \mathbb{R}_{[0,1)}$, $\mathcal{Z}$ is referred to as $\lambda-$contractive.*

Discrete-time model (3.5) assumes constant process noise between two consecutive samplings. This assumption, however, may provide poor discrete-time approximation of the continuous-time plant when a chosen sampling period does not correspond to rate of the process noise variation. By exploiting the inter-sampling architecture for NCS (see Section 2.2), this phenomenon is taken into account by the following discrete-time model (see (2.18)):

$$z[t_{k+1}] = (A - BK)z[t_k] + \sum_{i=0}^{N-1} \tilde{A}^i \tilde{E}\omega[t_k + (N-1-i)T], \tag{3.7}$$

where $A \in \mathbb{R}^{n\times n}$ is the state matrix with respect to the sampling period $T_s \in \mathbb{R}^+$, while $\tilde{A} \in \mathbb{R}^{n\times n}$, $\tilde{E} \in \mathbb{R}^{n\times p}$ are the matrices with respect to the inter-sampling period $T \in \mathbb{R}^+$, $T_s = NT$, $N \in \mathbb{Z}^+$ (for further details see Chapter 2). The additive disturbance $\omega[t_k + iT] \in \mathcal{W},\ \ \forall i \in \mathbb{Z}_{[0,N-1]}$, where $\mathcal{W} \subset \mathbb{R}^p$ is a bounded C-set. Notice that if $\omega[t_k] = \omega[t_k + T] = \ldots = \omega[t_k + (N-1)T]$, then (3.7) can be rewritten as the standard discrete-time state equation (3.5).

In order to define the robust positive invariance notion with respect to dynamics (3.7), one has to take into consideration variation of the process noise signal between two consecutive samplings (but constant between two consecutive inter-samplings).

Definition 3.20. *The $C-$set $\mathcal{Z} \subseteq \mathbb{R}^n$ is said to be robustly positively invariant (RPI) with respect to (3.7) if $z[t_{k+1}] \in \lambda\mathcal{Z}, \lambda \in \mathbb{R}_{[0,1]}$, for any $z[t_k] \in \mathcal{Z}$ and $\omega[t_k + iT] \in \mathcal{W},\ \ \forall i \in \mathbb{Z}_{[0,N-1]}$, i.e., if $(A - BK)\mathcal{Z} \oplus \left\{\bigoplus_{i=1}^{N-1} \tilde{A}^i \tilde{E}\mathcal{W}\right\} \subseteq \lambda\mathcal{Z}$. For $\lambda \in \mathbb{R}_{[0,1)}$, $\mathcal{Z}$ is referred to as $\lambda-$contractive.*

The problem of existence for invariant set with respect to (3.5) and (3.7) is determined by stability of the matrix $A - BK$. This well-known result from the control theory is reported without proof by the following lemma (for more details see e.g. Kolmanovsky and Gilbert [1998], Raković [2007]).

Lemma 3.2. *Let $\omega[t_k] \in \mathcal{W}$, where $\mathcal{W}$ is a C-set. The system (3.5) admits a non-trivial RPI set if $\rho(A - BK) < 1$.*

Remark 3.1. For a $C-$set $\mathcal{W}$, $\bigoplus_{i=1}^{N-1} \tilde{A}^i \tilde{E} \mathcal{W}$ is also a $C-$set. Hence, the previous lemma is applicable to (3.7) as well. The same holds for most of the results outlined in the sequel. Therefore, whenever we consider (3.5), unless it is suggested differently, same results apply also to (3.7).

The following result was proposed by Kofman et al. [2007] and it represents an effective method for computation of an invariant region with respect to (3.5) with diagonalizable and Schur matrix $A-BK$. An advantage of this result is that it provides a low complexity invariant region.

Theorem 3.1 (Kofman et al. [2007])**.** *Consider system (3.5) and let $A - BK = V\Lambda V^{-1}$ be the Jordan decomposition of the Schur matrix $A - BK$ with diagonal Λ and invertible V. Consider also a nonnegative vector $\bar{\omega}$ such that $|E\omega| \leq \bar{\omega}$, $\forall \omega \in \mathcal{W}$. For $\epsilon \in \mathbb{R}^n_{[0,\infty)}$, define*

$$\mathcal{S}(\epsilon) \left\{ z \in \mathbb{R}^n \ : \ \left| V^{-1} z \right| \leq (I_n - |\Lambda|)^{-1} \bar{\omega} + \epsilon \right\}. \tag{3.8}$$

Then:

- *For any $\epsilon \in \mathbb{R}^n_{[0,\infty)}$, the set $\mathcal{S}(\epsilon)$ is RPI.*
- *Given $\epsilon \in \mathbb{R}^n_{[0,\infty)}$ and $z[t_k] \in \mathbb{R}^n$, there exists $k^* \in \mathbb{Z}_{[k,\infty)}$ such that $z[t_k] \in \mathcal{S}(\epsilon)$ $\forall k \in \mathbb{Z}_{[k^*,\infty)}$.*

Remark 3.2. For $\epsilon \in \mathbb{R}_{(0,\infty)}$ the set $\mathcal{S}(\epsilon)$ is contractive. If, however, $\epsilon = 0$, the set is invariant, while the contractivity is not assured. Moreover, the shape of (3.8) is determined by the eigenvalues of $A - BK$. Namely, if $\lambda_i \in \mathbb{R}$ $\forall i \in \mathbb{Z}_{[1,n]}$, where $\lambda_i \in \sigma(A - BK)$, then (3.8) is a polytope. On the other side, if $\exists i \in \mathbb{Z}_{[1,n]}$ such that $\lambda_i \in \mathbb{C}$, then the resulting set is an intersection of ellipsoids or ellipsoids and parallel hyperplanes (see Kofman et al. [2007]).

For some applications, the RPI sets of the form (3.8) can be conservative. For instance, for performance analysis and synthesis of controllers for uncertain system (see Mayne et al. [2005]) or in the fault tolerant control design (see Olaru et al. [2010]), it is advantageous to use a RPI set which is as small as possible.

Definition 3.21. *The minimal robust positively invariant (mRPI) set with respect to (3.5) is defined as the RPI set contained in any closed RPI set. The mRPI set is unique, compact and contains the origin if $\mathcal{W}$ contains the origin.*

In order to elaborate a numerical method for construction of the mRPI set, we need to define first the *reachability set.*

Definition 3.22. *Given the set $\mathcal{X} \subset \mathbb{R}^n$ and the dynamics (3.5), the reachability set $\mathcal{R}_{t_k}(\mathcal{X})$ from $\mathcal{X}$ at the instance $k \in \mathbb{Z}_{[0,\infty)}$ is the set of all vectors x for which there exist $x[t_0] \in \mathcal{X}$ and $\bar{\omega}_{[t_{k-1}]}$ such that $x[t_k] = x$.*

Regarding (3.5), its $0-$*reachable* set at $t = t_k$ is determined according to

$$\mathcal{R}_{t_k}(\{0\}) = \mathcal{R}_{t_k} = \left\{ z = \sum_{i=0}^{k} (A - BK)^i E\omega[t_{k-i}] \ : \ \forall \omega[t_{k-i}] \in \mathcal{W} \right\}. \tag{3.9}$$

This relation can be also written in the space of sets as

$$\mathcal{R}_{t_k} = \bigoplus_{i=0}^{k} (A - BK)^i E\mathcal{W}, \tag{3.10}$$

where $\mathcal{R}_{t_{k-1}} \subseteq \mathcal{R}_{t_k}$[3]. Sets $\mathcal{R}_{t_k}$ are nested for each $k \in \mathbb{Z}_{[0,\infty)}$. Clearly, if $\mathcal{W}$ is a $C-$set, then $\mathcal{R}_{t_k}$ are also C-sets.

The mRPI set is defined by the following limit set:

$$\mathcal{R}_{t_\infty} = \lim_{k \to \infty} \mathcal{R}_{t_k}. \tag{3.11}$$

Even though $\mathcal{R}_{t_k}$ are $C-$sets for a finite k, this is not always true for the mRPI sets because they may not be closed (for an example see Blanchini and Miani [2008]). Notice that $\mathcal{R}_{t_k}$ is an inner approximation of the set $\mathcal{R}_{t_\infty}$. However, such an internal approximation is not particularly useful because, in general, it is not an invariant set.

Computation of an exact representation of the mRPI set is not possible in the general case, but only under restrictive assumptions such as when the matrix A is nilpotent Mayne and Schroeder [1997]. One then needs to resort to approximations, and different algorithms for the construction of RPI approximations can be found in the literature. Result in Raković et al. [2005] provides iterative approach which provides an approximation with arbitrary precision at the cost of an increased complexity. This result is outlined without proof in the following theorem.

Theorem 3.2 (Raković et al. [2005])**.** *If $0 \in \mathcal{W}$, then there exist a finite integer $s \in \mathbb{Z}_{(0,\infty)}$ and a scalar $\beta \in \mathbb{R}_{[0,1)}$ that satisfy*

$$(A - BK)^s \mathcal{W} \subseteq \beta \mathcal{W}. \tag{3.12}$$

Moreover,

$$\mathcal{R}(s, \beta) = (1 - \beta)^{-1} \mathcal{R}_{t_s} \tag{3.13}$$

[3]This inclusion holds if and only if $0 \in \mathcal{W}$.

is a convex and compact RPI set with respect to (3.5) *and it includes the origin in its interior. Furthermore,* $\mathcal{R}_{t_\infty} \subseteq \mathcal{R}(s, \beta)$.

The quality of an invariant approximation of the mRPI set is determined by s and β according to:

$$\begin{aligned} &\mathcal{R}(\underline{s}(\beta), \beta) \to \mathcal{R}_{t_\infty} \text{ as } \beta \to 0; \\ &\mathcal{R}(s, \underline{\beta}(s)) \to \mathcal{R}_{t_\infty} \text{ as } s \to \infty, \end{aligned} \tag{3.14}$$

where

$$\begin{aligned} \underline{s}(\beta) &= \min\{s \in \mathbb{Z}_{(0,\infty)} \ : \ (A - BK)^s \mathcal{W} \subseteq \beta \mathcal{W}\}; \\ \underline{\beta}(s) &= \min\{\beta \in \mathbb{R}_{[0,1)} \ : \ (A - BK)^s \mathcal{W} \subseteq \beta \mathcal{W}\}. \end{aligned}$$

For more details we refer to Raković et al. [2005]. An alternative method for computation of the mRPI set was proposed by Olaru et al. [2010] and it gives comparative result with the previous theorem regarding numerical complexity and precision of the approximations.

Apart from their general application in constrained control design, mRPI sets are also used for the characterization of the process and measurement noise effects on the closed-loop dynamics. Of course, instead of the mRPI sets, one may also use ultimate bounds computed according to Theorem 3.1. However, mRPI sets determine (by definition), the smallest possible invariant region in the state-space that one can possibly have. This is important because fault and delay detection mechanisms provided in Chapter 5 and Chapter 6 are based on separation of invariant sets which are obtained if one would use "healthy" or "faulty" feedback information for control. Therefore, by accepting certain degree of numerical complexity, mRPI sets yield the detection mechanism which is more sensitive to fault and delay occurrences. This is elaborated in more details in Chapter 5 and Chapter 6.

Regarding the dynamics (3.5) and (3.7), the corresponding mRPI sets are linked by the following result.

Proposition 3.3. *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be the mRPI sets for* (3.5) *and* (3.7), *respectively. Then* $\mathcal{X} \subseteq \mathcal{Y}$.

Proof. Denote by $\mathcal{R}^x_{t_k}$ and $\mathcal{R}^y_{t_k}$ $0-$reachable sets for (3.5) and (3.7) respectively. Since $(M + N)\mathcal{S} \subseteq M\mathcal{S} \oplus N\mathcal{S}$ for any $M \in \mathbb{R}^{n\times n}$, $N \in \mathbb{R}^{n\times n}$ and $\mathcal{S} \subset \mathbb{R}^n$ (see e.g. Ziegler [1995]), one can notice that

$$\left(\sum_{j=0}^{N-1} \tilde{A}^j \tilde{E}\right) \mathcal{W} \subseteq \bigoplus_{j=0}^{N-1} \tilde{A}^j \tilde{E}\mathcal{W}.$$

Here we used the fact that $E = \sum_{i=0}^{N-1} \tilde{A}^i \tilde{E}$ (see Section 2.2). Furthermore, this inclusion implies

$$\mathcal{R}^x_{t_k} \subseteq \mathcal{R}^y_{t_k}, \ \forall k \in \mathbb{Z}_{[0,\infty]},$$

where $\mathcal{R}^x_{t_k} = \bigoplus_{i=0}^{k} (A - BK)^i E\mathcal{W}$ and $\mathcal{R}^y_{t_k} = \bigoplus_{i=0}^{k} (A - BK)^i \left\{ \bigoplus_{j=0}^{N-1} \tilde{A}^j \tilde{E}\mathcal{W} \right\}$.

The proof is completed by observing that $\mathcal{X} = \mathcal{R}^x_{t_\infty}$ and $\mathcal{Y} = \mathcal{R}^y_{t_\infty}$. □

For control systems with multi-sensor configurations, where each sensor is possibly affected by measurement noise from a different bounded range, the closed-loop model is defined by:

$$z[t_{k+1}] = (A - BK)z[t_k] + E\omega_i[t_k], \quad i \in \mathbb{Z}_{[1,M]}. \tag{3.15}$$

In (3.15) M denotes a number of sensors and $\omega_i \in \mathcal{W}_i$, where $\mathcal{W}_i$ are possibly different $C-$sets.

Invariant sets related to system (3.15) belong to the class of *star shaped* sets (see Rubinov and Yagubov [1986]).

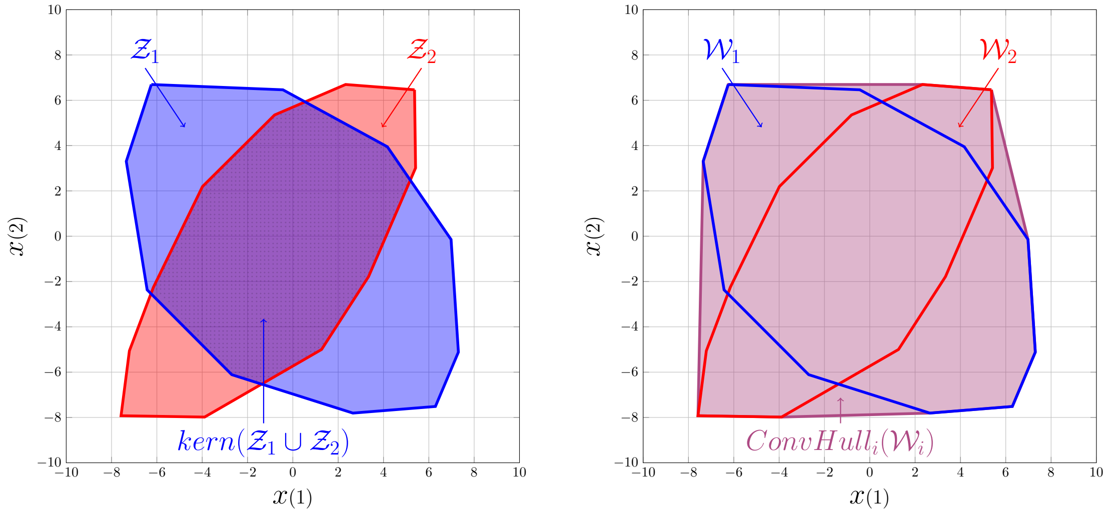
Definition 3.23. *A star shaped set $\mathcal{Z} \subset \mathbb{R}^n$ is a (connected and generally nonconvex) set for which exists a nonempty kernel:*

$$kern(\mathcal{Z}) = \{\bar{z} \in \mathcal{Z} \ : \ \bar{z} + \gamma(z - \bar{z}) \in \mathcal{Z}, \ \forall z \in \mathcal{Z}, \ \gamma \in \mathbb{R}_{[0,1]}\}. \tag{3.16}$$

A set is star shaped set at 0 if $0 \in kern(\mathcal{Z})$ (see Fig. 3.1a).

In order to characterize effects of switched disturbance sets on system (3.15), one has to consider union with respect to these sets. Therefore, system (3.15) admits a star shaped mRPI invariant set (see Olaru et al. [2010]). This stems from the fact that union of convex sets is a star shaped set. These non-convexity issues can be avoided by using the convex-hull of the noise bounding sets, i.e., we consider dynamics (3.5) where $\omega \in ConvHull(\mathcal{W}_i)$ $\forall i \in \mathbb{Z}_{[1,M]}$. This keeps the sets in a convex setting at the price of an increased conservatism (as illustrated in Fig. 3.1). With these tools and by using the boundedness assumptions for the process noises ω, it is possible to compute λ-contractive sets characterizing dynamics (3.5) according to one of the previously reported results.

For constrained control problem, i.e., for (3.3) such that $x \in \mathcal{X}$ and $u \in \mathcal{U}$, where $\mathcal{X}$ and $\mathcal{U}$ are $C-$sets as hard linear constraints on the state and control signal, instead of $0-$reachable sets, one can use a backward in time invariant set construction with respect to a given bounded region (see Glover and Schweppe [1971], Bertsekas and Rhodes [1971], Gilbert and Tan [1991]). This construction is briefly outlined in the sequel.



(A) Star-shaped set and its kernel

(B) Switching disturbance sets and the convex over-approximation

FIGURE 3.1: Illustration of set-theoretic notions.

Let us consider (3.3) with the linear state-feedback controller $u[t_k] = u_{ref}[t_k] + v[t_k]$, where $v = -K(x - x_{ref})$. Constraints on the state and the control action are given as

$$\begin{aligned} x &\in \{x_{ref}\} \oplus \bar{\mathcal{Z}}, \quad \bar{\mathcal{Z}} = \mathcal{P}(F_z, g_z) = \{z \in \mathbb{R}^n \ : \ F_z z \leq g_z\}, \\ v &\in \{u_{ref}\} \oplus \bar{\mathcal{V}}, \quad \bar{\mathcal{V}} = \mathcal{P}(F_v, g_v) = \{v \in \mathbb{R}^m \ : \ F_v v \leq g_v\}. \end{aligned} \tag{3.17}$$

For the considered control action, these constraints can be expressed as a function of z as

$$\mathcal{Z}_0 = \mathcal{P}(F_0, g_0) = \{z \in \mathbb{R}^n \ : \ F_0 x \leq g_0\}, \quad F_0 = \begin{bmatrix} F_z \\ -F_v K \end{bmatrix}, \quad g_0 = \begin{bmatrix} g_z \\ g_v \end{bmatrix}. \tag{3.18}$$

Assume that $z[t_{k+1}] \in \bar{\mathcal{Z}}_0$. Then,

$$\mathcal{Z}_1 = \left\{ z \in \mathbb{R}^n \ : \ \begin{bmatrix} F_0 \\ F_0(A - BK) \end{bmatrix} z \leq \begin{bmatrix} g_0 \\ g_0 - \max\limits_{\omega \in \mathcal{W}} \{F_0 E \omega\} \end{bmatrix} \right\}, \tag{3.19}$$

defines the set of *one step admissible states*, i.e., the set of all $z[t_k] \in \mathcal{Z}_0$ such that $z[t_{k+1}] \in \mathcal{Z}_0$. Similarly, further set iterations determine sets of admissible states with respect to $k \in \mathbb{Z}_{[2,\infty]}$ steps. These sets are computed in an iterative way as

$$\mathcal{Z}_k = \left\{ z \in \mathbb{R}^n \ : \ \begin{bmatrix} F_{k-1} \\ F_{k-1}(A - BK) \end{bmatrix} z \leq \begin{bmatrix} g_{k-1} \\ g_{k-1} - \max\limits_{\omega \in \mathcal{W}} \{F_{k-1}E\omega\} \end{bmatrix} \right\}. \tag{3.20}$$

Notice that $\mathcal{Z}_k \subseteq \mathcal{Z}_{k-1} \ \forall k \in \mathbb{Z}^+$.

The following definition is important in specifying the limit set, i.e., when number of iterations $k \to \infty$.

Definition 3.24. *Given a bounded region $\mathcal{X} \subset \mathbb{R}^n$, the maximal robust positively invariant (MRPI) set with respect to (3.5) is defined as the RPI set that contains all the RPI sets contained in $\mathcal{X}$.*

If there exists a finite $k \in \mathbb{Z}^+$ such that $\mathcal{Z}_k = \mathcal{Z}_{k-1}$, then $\mathcal{Z}_k = \mathcal{Z}_\infty$, where $\mathcal{Z}_\infty$ denotes the maximal robust positively invariant (MRPI) set in $\bar{\mathcal{Z}}$. However, there is still one important question regarding the MRPI set that requires the answer: for a given dynamics, when the MRPI set is determined in a finite number of iterations. For dynamics (3.5), the answer to this question is provided by Kolmanovsky and Gilbert [1998].

Theorem 3.3. *Consider $z[t_{k+1}] = (A - BK)z[t_k] + E\omega[t_k], \ \ \omega[t_k] \in \mathcal{W} \ \forall k \in \mathbb{Z}_+$, where $\mathcal{W}$ is a $C-$set. Let $\rho(A - BK) < 1$ and $\mathcal{R}_{t_\infty}$ is the corresponding mRPI set. If $\mathcal{R}_{t_\infty} \subset \mathcal{Z}_0$ then there exists a finite $k \in \mathbb{Z}^+$ such that $\mathcal{Z}_k = \mathcal{Z}_{k-1} = \mathcal{Z}_\infty$.*

We refer to Gilbert and Tan [1991], Kolmanovsky and Gilbert [1998], Nguyen [2012] for more information on backward computation of invariant sets.

3.3 Positive invariance for dDDE

Having a process modeled by LTI differential equation and controlled over a network, the closed-loop system can be described by linear discrete-time delay-difference equation (dDDE) with time-varying parameters:

$$x[t_{k+1}] = Ax[t_k] + \sum_{j=0}^{d_m} B_{d_m-j}(\tau_k)u[t_{k+j-d_m}] + E\omega[t_k].$$

In the special case when the network induces constant delays, the same system admits a representation with constant parameters:

$$x[t_{k+1}] = Ax[t_k] + \sum_{j=0}^{d_m} B_{d_m-j}u[t_{k+j-d_m}] + E\omega[t_k].$$

For more details see Section 2.2.

Stability analysis of the discrete-time delay difference equations is performed in the simplest way by introducing a new state vector which consists of the initial state vector and all input signals on the delay window (see e.g. Aström and Wittenmark [1997], Hetel et al. [2008]). Thus the obtained augmented state-space representation is linear and its stability can be (conceivably) assessed either by using Lyapunov direct method or by verifying (joint) spectral radius of the closed-loop system matrix (see Section 2.3.2 and Section 2.3.3). The same methodology can be applied for control design (see e.g. Chae et al. [2010]).

For the sake of simplicity, let us consider the following closed-loop dynamics for a state-feedback controller with neglected parametric uncertainties:

$$x[t_{k+1}] = \sum_{i=0}^{d_m} A_i x[t_{k-i}] + E\omega[t_k]. \tag{3.21}$$

When it comes to positive invariance, one may have tendency to apply the same logic (augmenting the state-space) as for the stability analysis and control design. As it was already mentioned in Chapter 2, for any finite delay realization, an extended state-space representation can be constructed by taking $X[t_k] = \begin{bmatrix} x[t_k]^T & \dots & x[t_{k-d_m}]^T \end{bmatrix}^T$. Then, (3.21) can be rewritten as:

$$X[t_{k+1}] = \mathbb{A}X[t_k] + \mathbb{E}\omega[t_k] = \begin{bmatrix} A_0 & \dots & A_{d_{m-1}} & A_{d_m} \\ I_n & \dots & 0_{n\times n} & 0_{n\times n} \\ \vdots & \ddots & \vdots & \vdots \\ 0_{n\times n} & \dots & I_n & 0_{n\times n} \end{bmatrix} X[t_k] + \begin{bmatrix} E \\ 0_{n\times p} \\ \vdots \\ 0_{n\times p} \end{bmatrix} \omega[t_k], \tag{3.22}$$

where the dimension and the structure of $\mathbb{A}$ depends on delay value. With an appropriate re-ordering of the indices, and by introducing $A_i = 0$ where necessary, we can obtain an extended state-space representation of (3.21) for any delay realization.

Definition 3.25. *The $C-$set $\mathcal{X} \subset \mathbb{R}^{n(d_m+1)}$ is said to be robustly positively invariant with respect to the augmented state-space representation* (3.22) *if $X[t_{k+1}] \in \lambda\mathcal{X}$, $\lambda \in \mathbb{R}_{[0,1]}$, for any $X[t_k] \in \lambda\mathcal{X}$ and $\omega \in \mathcal{W}$, i.e., $\mathbb{A}\mathcal{X} \oplus \mathbb{E}\mathcal{W} \subset \lambda\mathcal{X}$. For $\lambda\mathbb{R}_{[0,1)}$, $\mathcal{X}$ is referred to as $\lambda-$contractive with respect to* (3.22).

This invariance notion is often referred to as *augmented invariance* or invariance based on *Lyapunov-Krasovskii* approach (see Gielen [2013]).

It is worth mentioning that the fixed points and the periodic orbits are natural (and trivial) invariant sets. As such, for the linear dDDEs, the invariant sets contain the origin, and often the construction of non-trivial invariant sets starts from the assumption of having the origin as an interior point. For $\lambda-$contractiveness, the origin needs to be an interior point of $\mathcal{X}$ in order to have a proper set construction with an inclusion relationship.

However, positive invariance as outlined in Definition 3.25 needs to be taken into account carefully because it leads to a non-standard definition of positive invariance for time-delay systems. For this reason, it will be probably useful here to wonder: what is the definition of positive invariance for delay-difference equations? The standard invariance definition for systems without aftereffects states "...for any initial condition from the set...". Following the same logic, we may come out with the equivalent statement.

Definition 3.26. *The $C-$set $\mathcal{X} \subset \mathbb{R}^n$ is said to be robustly positively invariant set with respect to (3.21) if $x[t_{k+1}] \in \lambda\mathcal{X},\ \lambda \in \mathbb{R}_{[0,1]}$ for any initial condition from $\mathcal{X}$, i.e., for any $x[t_{k-i}] \in \mathcal{X},\ i \in \mathbb{Z}_{[0,d_m]}$ and for any $\omega \in \mathcal{W}$, i.e., $\bigoplus_{i=0}^{d_m} A_i\mathcal{X} \oplus E\mathcal{W} \subseteq \lambda\mathcal{X}$. For $\lambda \in \mathbb{R}_{[0,1)}$, $\mathcal{X}$ is defined as $\lambda-$contractive.*

In order to have better insight into this definition, let us consider the following simple scalar system:

$$x[t_{k+1}] = x[t_k] - 0.8x[t_{k-1}]. \tag{3.23}$$

According to Definition 3.26, invariant set for this system, say $\mathcal{X}$, would be defined as:

$$x[t_{k+1}] \in \mathcal{X}, \quad \forall x[t_k] \in \mathcal{X}, \quad \forall x[t_{k-1}] \in \mathcal{X}. \tag{3.24}$$

Regarding the augmented state-space representation, condition (3.24) can be rewritten as:

$$\begin{bmatrix} x[t_{k+1}] \\ x[t_k] \end{bmatrix} = \mathcal{A} \begin{bmatrix} x[t_k] \\ x[t_{k-1}] \end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix} 1 & -0.8 \\ 0 & 1 \end{bmatrix}, \tag{3.25}$$

i.e., as

$$\mathcal{A}\mathcal{X}^2 \subseteq \lambda\mathcal{X}^2. \tag{3.26}$$

Despite the fact that $\mathcal{A}$ is Schur, there does not exist a set $\mathcal{X}$ that satisfies the condition (3.26). This is easy to observe since $\|\mathcal{A}\|_\infty > 1$. However, dynamics (3.25) do admit an invariant set (see Fig. 3.2) defined according to Definition 3.25. By analyzing (3.23) and the invariant set shown in Fig. 3.2, one can notice that this set engenders new constraints on initial conditions, i.e., the initial conditions became coupled. This loss of "one degree of freedom" is compensated by the existence of invariant region which prohibits certain realizations of initial conditions.

From this consideration, it should be clear that positive invariance as proposed in Definition 3.26 is a stronger property than stability of (3.23), even though for (3.25) these notions are equivalent. Therefore, we refer to positive invariance defined in Definition 3.26 as $D-$*invariance* (see Lombardi et al. [2011a]). While the concept of positive invariance for the augmented state-space representations, i.e., for LTI systems with parametric uncertainties is already well-established in the literature (see e.g. Blanchini and Miani

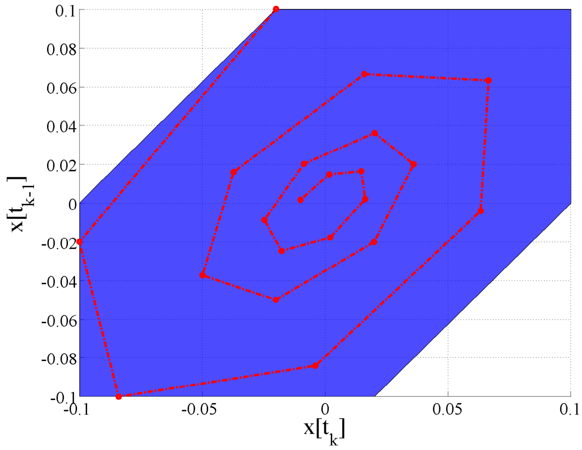


FIGURE 3.2: Positively invariant set with respect to (3.23)

[2008],Blanchini [1999] for backward in time computation and Barmish and Sankaran [1979] for the forward in time propagation for uncertain linear systems), in the rest of this chapter, we put the accent on $D-$invariance.

First results on D-invariance appeared in the work of Dambrine et al. [1995a] and Dambrine et al. [1995b]. The authors defined the $D-$invariant sets motivated by constrained control design problem for systems with delays. Several years later, the same sets were used by Hennet and Tarbouriech [1997] and Hennet and Tarbouriech [1998] to establish robust (with respect to delay parameter) stability condition for delay-differential and delay-difference equations, respectively. Recently, D-invariant sets regained attention in the works of Lombardi [2011], for dDDE, and Gielen [2013], for discrete-time delay difference inclusions (DDI).

Despite great advancements toward characterization of $D-$invariant sets, some fairly basic results are still missing, for instance the existence of $D-$invariant sets is not fully characterized. In this work we do not provide a complete answer to this problem either, but we do make some important steps ahead (see Chapter 4). In the remaining of this chapter we address the problem of existence and construction of the minimal robust $D-$invariant (mRDI) set for dDDEs affected by additive disturbance. We also provide some new concepts of positive invariance for delay-difference equations such as cyclic invariance, invariant families of sets and parametrized invariant sets. Consideration on $D-$invariance notion for delay-difference inclusions is not considered here and for more details an interested reader is referred to Gielen et al. [2012a], Gielen [2013].

3.4 Minimal robust D-invariant set

Consider the following closed-loop dynamics:

$$x[t_{k+1}] = \sum_{i=0}^{d_m} A_i x[t_{k-i}] + \omega[t_k], \tag{3.27}$$

where $\omega \in \mathcal{W}$, with $\mathcal{W} \subset \mathbb{R}^n$ as a $C-$set.

Lemma 3.3. *Denote by $\mathcal{I} = \{0, 1, \dots, d_m\}$ and let a symmetric $C-$set $\mathcal{X} \subset \mathbb{R}^n$ satisfy the following inclusion:*

$$\bigoplus_{i\in\mathcal{I}} A_i\mathcal{X} \subseteq \lambda\mathcal{X}, \quad \lambda \in \mathbb{R}_{[0,1)}.$$

For any $C-$set $\mathcal{W}$ and $\forall\varepsilon \in \mathbb{R}^+$, $\exists s \in \mathbb{Z}_{[1,\infty)}$ such that

$$\bigoplus_{c_s\in\mathcal{I}^s} \Pi_{c_s}\mathcal{W} \subseteq \varepsilon\mathcal{W}, \quad \Pi_{c_s} = A_{i_1}A_{i_2}\dots A_{i_s},\ i_j \in \mathcal{I},\ j \in \mathbb{Z}_{[1,s]}. \tag{3.28}$$

Proof. Assume that the statement of the lemma holds true. Consider the set map $\mathcal{R}(\mathcal{X}) = \bigoplus_{i\in\mathcal{I}} A_i\mathcal{X}$. The subsequent iterate of the map for $\mathcal{R}(\mathcal{X})$ is determined as:

$$\mathcal{R}(\mathcal{R}(\mathcal{X})) = \mathcal{R}^2(\mathcal{X}) = \bigoplus_{i\in\mathcal{I}} A_i\mathcal{R}(\mathcal{X}) = \bigoplus_{c_2\in\mathcal{I}^2} \Pi_{c_2}\mathcal{X}.$$

Since $\mathcal{R}(\mathcal{X}) \subseteq \lambda\mathcal{X}$, by using Lemma 3.1 one can show that

$$\mathcal{R}^2(\mathcal{X}) = \bigoplus_{i\in\mathcal{I}} A_i\mathcal{R}(\mathcal{X}) \subseteq \bigoplus_{i\in\mathcal{I}} A_i\lambda\mathcal{X} \subseteq \lambda^2\mathcal{X}.$$

Assume that $\mathcal{R}^k(\mathcal{X}) \subseteq \lambda^k\mathcal{X}$. Then

$$\mathcal{R}^{k+1}(\mathcal{X}) = \bigoplus_{i\in\mathcal{I}} A_i\mathcal{R}^k(\mathcal{X}) \subseteq \bigoplus_{i\in\mathcal{I}} A_i\lambda^k\mathcal{X} \subseteq \lambda^{k+1}\mathcal{X}.$$

Since $\alpha \bigoplus_{i\in\mathcal{I}} A_i\mathcal{X} \subseteq \alpha\lambda\mathcal{X}$, then for any $C-$set $\mathcal{W}$ and $\forall\varepsilon \in \mathbb{R}^+$, there $\exists\alpha_1, \alpha_2 \in \mathbb{R}^+$ such that $\alpha_1\mathcal{X} \subseteq \epsilon\mathcal{W} \subseteq \alpha_2\mathcal{X}$. This implies that there $\exists s \in \mathbb{Z}^+$, $\lambda^s\alpha_2 < \alpha_1$ such that:

$$\bigoplus_{c_s\in\mathcal{I}^s} \Pi_{c_s}\alpha_2\mathcal{X} \subseteq \lambda^s\alpha_2\mathcal{X} \subseteq \alpha_1\mathcal{X} \subseteq \epsilon\mathcal{W}.$$

□

Remark 3.3. For the previous results we assumed the existence of a $\lambda-$contractive set $\mathcal{X}$ with respect to the nominal (without additive disturbance) part of the system (3.27). The existence of such a contractive set is an open problem in the literature. However, in the following chapter we provide some important discussion that concerns the existence of these sets for a given dynamics.

The following well-known result (see e.g. Khamsi and Kirk [2011]) will be used later to prove the existence of the mRDI set.

Lemma 3.4. *Let $(\mathcal{X}, d(\cdot,\cdot))$ be a complete metric space and let $f(\cdot) : \mathcal{X} \to \mathcal{X}$ be a contractive function with contraction factor $\lambda \in [0,1)$ that is*

$$d(f(x), f(y)) \leq \lambda d(x,y), \quad \lambda \in [0,1)$$

holds for all $x, y \in \mathcal{X}$. If $\mathcal{X}$ is a compact set, then there exists a unique fixed point $\bar{x} \in \mathcal{X}$ such that $f(\bar{x}) = \bar{x}$.

Equivalent formulation of the Banach contraction principle can be also applied to sets. As a measure of distance between two sets, we use the *Hausdorf distance* which is given as:

$$H_{\mathcal{X}}(\mathcal{Y}, \mathcal{Z}) = \min_{\alpha} \{\alpha \ : \ \mathcal{Y} \subseteq \mathcal{Z} \oplus \alpha\mathcal{X}, \ \mathcal{Z} \subseteq \mathcal{Y} \oplus \alpha\mathcal{X}, \ \alpha \in \mathbb{R}_+\}, \tag{3.29}$$

where $\mathcal{X}$ is a symmetric $C-$set.

Consider (3.27) and let define the set-valued map $\mathcal{R}_w(\mathcal{Y}) = \bigoplus\limits_{i\in\mathcal{I}} A_i\mathcal{Y} \oplus \mathcal{W}$. The following theorem, which uses the Banach contraction principle, provides the basic result in stating the sufficient condition for the existence of the mRDI set. This theorem is similar with the theorem proposed by Raković [2007] on the existence of the mRPI set for discrete-time linear systems.

Theorem 3.4. *Let there exist a symmetric $C-$set $\mathcal{X} \subset \mathbb{R}^n$ such that*

$$\bigoplus_{i\in\mathcal{I}} A_i\mathcal{X} \subseteq \lambda\mathcal{X}, \quad \lambda \in \mathbb{R}_{[0,1)}.$$

Then, $H_{\mathcal{X}}\left(\mathcal{R}_w(\mathcal{Y}), \mathcal{R}_w(\mathcal{Z})\right) \leq \lambda H_{\mathcal{X}}\left(\mathcal{Y}, \mathcal{Z}\right)$ holds for any $C-$sets $\mathcal{Y} \subset \mathbb{R}^n$ and $\mathcal{Z} \subset \mathbb{R}^n$.

Proof. Let denote by $\mathcal{Y}$ and $\mathcal{Z}$ two arbitrary C-sets and $H_{\mathcal{X}}(\mathcal{Y}, \mathcal{Z}) = \alpha$. From the definition of the Hausdorff distance we have that:

$$\mathcal{Y} \subseteq \mathcal{Z} \oplus \alpha\mathcal{X}, \qquad \mathcal{Z} \subseteq \mathcal{Y} \oplus \alpha\mathcal{X}.$$

By multiplying both sides of the previous inclusions by $A_i, \ i \in \mathcal{I}$ and by summing them up (see Lemma 3.1) we have

$$\bigoplus_{i\in\mathcal{I}} A_i\mathcal{Y} \subseteq \bigoplus_{i\in\mathcal{I}} A_i\mathcal{Z} \oplus \alpha \bigoplus_{i\in\mathcal{I}} A_i\mathcal{X}$$
$$\bigoplus_{i\in\mathcal{I}} A_i\mathcal{Z} \subseteq \bigoplus_{i\in\mathcal{I}} A_i\mathcal{Y} \oplus \alpha \bigoplus_{i\in\mathcal{I}} A_i\mathcal{X}$$

From the statement of the theorem and by adding to the both sides $\mathcal{W}$, it follows that

$$\begin{aligned} \mathcal{R}_w(\mathcal{Y}) &\subseteq \mathcal{R}_w(\mathcal{Z}) \oplus \alpha\lambda\mathcal{X}, \\ \mathcal{R}_w(\mathcal{Z}) &\subseteq \mathcal{R}_w(\mathcal{Y}) \oplus \alpha\lambda\mathcal{X}, \end{aligned} \tag{3.30}$$

which is equal to $H_{\mathcal{X}}\left(\mathcal{R}_w(\mathcal{Y}), \mathcal{R}_w(\mathcal{Z})\right) \leq \lambda\alpha$ that is $H_{\mathcal{X}}\left(\mathcal{R}_w(\mathcal{Y}), \mathcal{R}_w(\mathcal{Z})\right) \leq \lambda H_{\mathcal{X}}(\mathcal{Y}, \mathcal{Z})$. □

Corollary 3.1. *Denote by $\mathcal{R}_w(\{0\}) = \mathcal{R}_w = \mathcal{W}$ and let $\mathcal{R}_w^k$ is the k^{th} iterate of the set-valued map with respect to (3.27) ($0-$reachable set in k steps). Since the previous theorem showed that the map $\mathcal{R}_w(\mathcal{Y})$ is a contraction for any compact set $\mathcal{Y}\mathbb{R}^n$, the iterates $\mathcal{R}_w^k(\mathcal{Y})$ converge to $\mathcal{R}_w^\infty$ as $k \to \infty$, where $\mathcal{R}_w^\infty$ is an attractor with basin of attraction being the whole space, i.e., $\mathcal{R}_w^\infty$ is the minimal robust D-invariant set.*

For smaller delays and small number of states, it is possible to construct an invariant approximation of the mRDI set by using $0-$reachable sets. In order to show this construction, let us introduce the following sets of indices: $\mathcal{I}^0 = \{-1\}$, where $A_{-1} = I_n$ and $\mathbb{I}^k = \{\mathcal{I}^k, \mathcal{I}^{k-1}, \ldots, \mathcal{I}^0\}$, with $\mathcal{I} = \{0, 1, \ldots, d_m\}$. By using these notations, we can represent the $0-$reachable set for the k^{th} step as:

$$\mathcal{R}_w^k = \bigoplus_{c_k \in \mathbb{I}^k} \Pi_{c_k}\mathcal{W}. \tag{3.31}$$

Invariant approximations of the mRDI set are constructed according to the following theorem.

Theorem 3.5. *Let assume that for each $\varepsilon \in \mathbb{R}^+$, $\exists s \in \mathbb{Z}^+$ such that*

$$\bigoplus_{i \in \mathcal{I}^{s+1}} A_i\mathcal{W} \subseteq \varepsilon\mathcal{W}. \tag{3.32}$$

Then

$$\mathcal{R}_w(s, \varepsilon) = (1-\varepsilon)^{-1}\mathcal{R}_w^s \tag{3.33}$$

is an invariant approximation of the mRDI set such that $\mathcal{R}_w^\infty \subseteq \mathcal{R}_w(s, \varepsilon)$ which is a $C-$set.

Proof. The property of invariant approximations of the mRDI set being $C-$sets follows from the fact that Minkowski sum and scaling of a $C-$set is also a $C-$set. When (3.32) holds, we aim to show that $\mathcal{R}_w(s, \varepsilon)$ is a D-invariant set, with the same arguments as in the LTI case (see Raković et al. [2005]).

$$\begin{aligned}
\bigoplus_{i\in\mathcal{I}} A_i\mathcal{R}_w(s,\varepsilon)\oplus\mathcal{W} &= (1-\varepsilon)^{-1}\left\{\bigoplus_{i\in\mathcal{I}} A_i\mathcal{R}_w^s\right\}\oplus\mathcal{W}\\
&= (1-\varepsilon)^{-1}\left\{\bigoplus_{i\in\mathcal{I}} A_i\left\{\bigoplus_{c_s\in\mathbb{I}^s}\Pi_{c_s}\mathcal{W}\right\}\right\}\oplus\mathcal{W}\\
&= (1-\varepsilon)^{-1}\left\{\bigoplus_{i\in\mathcal{I}^{s+1}} A_i\mathcal{W}\oplus\bigoplus_{c_s\in\mathbb{I}^s\setminus\mathcal{I}^0}\Pi_{c_s}\mathcal{W}\right\}\oplus\mathcal{W}\\
&\subseteq (1-\varepsilon)^{-1}\left\{\varepsilon\mathcal{W}\oplus\bigoplus_{c_s\in\mathbb{I}^s\setminus\mathcal{I}^0}\Pi_{c_s}\mathcal{W}\right\}\oplus\mathcal{W}\\
&= (1-\varepsilon)^{-1}\left\{\bigoplus_{c_s\in\mathbb{I}^s\setminus\mathcal{I}^0}\Pi_{c_s}\mathcal{W}\right\}\oplus(\varepsilon(1-\varepsilon)+1)\mathcal{W}\\
&= (1-\varepsilon)^{-1}\left\{\bigoplus_{c_s\in\mathbb{I}^s}\Pi_{c_s}\mathcal{W}\right\} = \mathcal{R}_w(s,\varepsilon).
\end{aligned}$$

By definition, the mRDI set is included in any RDI set. Thus, $\mathcal{R}_w(s,\varepsilon)$ is an outer invariant approximation of the mRDI set. □

Quality of an approximation is determined in the same way as in (3.14).

Direct application of the exposed theorems via Minkowski sum requires high computational effort and therefore is only applicable to systems with small number of states and small delays. This can be mitigated using an approach based on the half-space representation of polyhedral sets. For further details on this approach we refer to Raković et al. [2005].

3.5 Positive invariance for dDDE. Alternatives and further advancements

The section establishes a new perspective on the positively invariant sets construction for discrete-time delay-difference equations and their structural properties via set factorization.

For the sake of simplicity, let us consider the following closed-loop dynamics by neglecting parametric uncertainties and additive disturbance:

$$x[t_{k+1}] = \sum_{i=0}^{d_m} A_i x[t_{k-i}]. \tag{3.34}$$

In analogy with the invariance notions introduced in Section 3.2, the following concept of cyclic invariance (see Lombardi et al. [2012]) can offer a certain degree of flexibility by proposing a family of invariant sets instead of a rigid object in $(\mathbb{R}^n)^{d_m}$ or $\mathbb{R}^n$ as introduced in Definition 3.25 and Definition 3.26.

Definition 3.27. *A family of ($d_m + 1$ tuples of) sets $\{\mathcal{X}_0, \dots, \mathcal{X}_{d_m}\}$ is called cyclic D-invariant with respect to (3.34) if:*

$$\begin{aligned}
&A_0\mathcal{X}_0 \oplus A_1\mathcal{X}_1 \oplus \cdots \oplus A_{d_m}\mathcal{X}_{d_m} \subseteq \mathcal{X}_{d_m};\\
&A_0\mathcal{X}_{d_m} \oplus A_1\mathcal{X}_0 \oplus \cdots \oplus A_{d_m}\mathcal{X}_{d_m-1} \subseteq \mathcal{X}_{d_m-1};\\
&\qquad\qquad\qquad\vdots\\
&A_0\mathcal{X}_1 \oplus A_1\mathcal{X}_2 \oplus \cdots \oplus A_{d_m}\mathcal{X}_0 \subseteq \mathcal{X}_0.
\end{aligned} \tag{3.35}$$

Another definition of the positive invariance which generalizes the cyclic invariance to a invariant family of sets, was proposed by Rakovic et al. [2012].

Definition 3.28. *A family of ($d_m + 1$ tuples) of sets $\mathcal{F} \subset (\mathbb{R}^n)^{d_m+1}$ is an invariant family with respect to (3.34) if for all $\{\mathcal{X}_0, \mathcal{X}_1, \dots, \mathcal{X}_{d_m}\} \in \mathcal{F}$, there exists a set $\mathcal{X} \subset \mathbb{R}^n$ such that $\{\mathcal{X}, \mathcal{X}_0 \dots, \mathcal{X}_{d_m-1}\} \in \mathcal{F}$ and*

$$A_0\mathcal{X}_0 \oplus A_1\mathcal{X}_1 \oplus \cdots \oplus A_{d_m}\mathcal{X}_{d_m} \subseteq \mathcal{X}. \tag{3.36}$$

In the subsequent part, we present several geometrical notions related to set factorization which allow establishing a novel connection between extended and D-invariance and even to propose new invariant set structures.

Given a set of indices $I \subset \mathbb{Z}_{[1,m]}$, a partition of I is described as $I = \bigcup_{k=1}^{l} = I_k$ with $I_i \cap I_j = \emptyset$. Given a subset $I_i \subset \mathbb{Z}_{[1,m]}$ and a set $P \in \mathbb{R}^m$, we denote $P_{\downarrow I_i}$ the projection of the set P on the subset of $\mathbb{R}^n$ with indices of Cartesian coordinates in I_i.

Definition 3.29. *Let a set $\mathcal{X} \in \mathbb{R}^m$ and a partition $\mathbb{Z}_{[1,m]} = \bigcup_{i=1}^{l} = I_i$.*

1. *The $\mathcal{X}$* can be factored *according to the partition $\mathbb{Z}_{[1,m]} = \bigcup_{i=1}^{l} = I_i$ if:*

$$\mathcal{X} = \mathcal{X}_{\downarrow I_1} \times \cdots \times \mathcal{X}_{\downarrow I_l} \tag{3.37}$$

2. *A* set factorization (3.37) *is* balanced *if* $card\{I_1\} = \cdots = card\{I_l\}$.

3. *An* ordered factorization *is defined by an* ordered *partition* $I = \bigcup_{k=1}^{l} = I_k$ *which satisfies*

$$max\{I_i\} < min\{I_j\}, \forall i < j; \tag{3.38}$$

4. *A* regular factorization *is characterized by the equivalence of the factors*

$$\mathcal{X}_{\downarrow I_1} = \cdots = \mathcal{X}_{\downarrow I_l} = P \tag{3.39}$$

and leads to a relationship:

$$\mathcal{X} = \underbrace{P \times P \times \cdots \times P}_{l\ times} \tag{3.40}$$

Several remarks can be done with respect to the previous definitions:

- The set factorization is a non-commutative set operation (a Cartesian product inherited property). The regular factorization is one of the special cases where such commutativity property holds inside the given partition.
- Any regular factorization is balanced.
- The regularity of a factorization does not imply the ordering of the partition.
- A balanced and ordered factorization is not necessarily regular.

3.5.1 Polyhedral set factorization

The general definitions of factorization do not build on specific geometry of the set $\mathcal{X}$. Most of the properties are related to the Cartesian product operation. However, it is worth to be mentioned that the geometry of the factors can be related to the geometry of the set $\mathcal{X}$. For example, the convexity of the set $\mathcal{X}$ implies the convexity of the factors and the polyhedral structure of factors implies a polyhedral structure for the set $\mathcal{X}$. Noteworthy, even if the projection of ellipsoidal set is ellipsoidal, one cannot expect to obtain factorizations of ellipsoidal set as long as the Cartesian product of ellipsoids is not an ellipsoid.

From these remarks it becomes clear that the polyhedral sets represent a particularly interesting class which can be used as a framework for the developments in relationship with set factorization. We point the reader to the references Halbwachs et al. [2003], Halbwachs et al. [2006] and recall here a property of the polyhedral factorization:

Proposition 3.4. *The ordered factorization of a polyhedral set described in its (non-redundant) half-space representation*

$$\mathcal{X} = \{x \in \mathbb{R}^n | Fx \leq w\} \tag{3.41}$$

is related to a block diagonalization of the matrix F according to a column permutation given by the partition of indices $\mathbb{Z}_{[1,n]} = \bigcup_{i=1}^{l} I_i$.

Before continuing to the relationship between set factorization and positive invariance, we note that the *minimal/maximal* factorization for any polyhedron P is defined in terms of the number of factors. The *minimal* factorization is defined by the trivial partition $I_1 = \mathbb{Z}_{[1,n]}$.

3.5.2 Invariance in augmented state-space and $D-$invariance

Starting from the definitions of the invariance in the extended state space and the D-invariance we can observe that the dimension of the respective state space allows a regular factorization type of relationship. This relationship is formally stated in the next theorem.

Theorem 3.6. *The system (3.34) admits a non-trivial*[4] *D-invariant set if and only if there exists an invariant set for*

$$X[t_{k+1}] = \begin{bmatrix} A_0 & \dots & A_{d_{m-1}} & A_{d_m} \\ I_n & \dots & 0_{n\times n} & 0_{n\times n} \\ \vdots & \ddots & \vdots & \vdots \\ 0_{n\times n} & \dots & I_n & 0_{n\times n} \end{bmatrix} X[t_k], \tag{3.42}$$

which admits a regular ordered factorization.

Proof. The necessity can be proved by observing that the existence of a $D-$invariant set P with respect to (3.34) implies that for $x[t_{k-i}] \in P$, $i \in \mathbb{Z}_{[0,d_m]}$ then $x[t_k] \in P$. Consequently, for any extended vector $X[t_k] = \left[x[t_k]^T \dots x[t_{k-d_m}]^T\right]^T$ contained in $\mathcal{X} = P \times P \times \dots \times P = P^{d_m+1}$ it holds that $X[t_{k+1}] \in \mathcal{X}$, which ensures the positive invariance of $\mathcal{X}$ with respect to (3.42).

Proving the sufficiency is slightly more involved. The regular order factorization implies the existence of a set $P = \mathcal{X}_{\downarrow \mathbb{Z}_{[1,n]}} = \mathcal{X}_{\downarrow \mathbb{Z}_{[n+1,2n]}} = \dots = \mathcal{X}_{\downarrow \mathbb{Z}_{[nd_m+1,n(d_m+1)]}}$. In order to

[4] A nontrivial invariant set is understood as being non-degenerate, compact, convex and containing the origin in its interior.

prove its $D-$invariance we write the first line of the extended dynamics (3.42) as:

$$x[t_{k+1}] = \begin{bmatrix} A_0 & \dots & A_{d_m-1} & A_{d_m} \end{bmatrix} X[t_k], \tag{3.43}$$

By expressing the vector in the extended form $X[t_k] = \begin{bmatrix} x[t_k]^T & \dots x[t_{k-d_m}]^T \end{bmatrix}^T$ and exploiting the factorization, we get a series of relations starting from the invariance property:

$$X[t_k] \in P \to x[t_{k+1}] \in \mathcal{X}_{\downarrow \mathbb{Z}_{[1,n]}} \tag{3.44}$$

is equivalent with:

$$\begin{bmatrix} A_0 & \dots & A_{d_m-1} & A_{d_m} \end{bmatrix} X[t_{k-1}] \in \mathcal{X}_{\downarrow \mathbb{Z}_{[1,n]}}, \quad \forall X[t_k] \in \mathcal{X} \tag{3.45}$$

or set-wise:

$$\begin{bmatrix} A_0 & \dots & A_{d_m-1} & A_{d_m} \end{bmatrix} \mathcal{X} \subseteq \mathcal{X}_{\downarrow \mathbb{Z}_{[1,n]}} \tag{3.46}$$

Replacing the extended set by its factorized formulation we get:

$$\begin{bmatrix} A_0 & \dots & A_{d_m-1} & A_{d_m} \end{bmatrix} \left(\mathcal{X}_{\downarrow \mathbb{Z}_{[1,n]}} \times \dots \times \mathcal{X}_{\downarrow \mathbb{Z}_{[d_m n+1,(d_m+1)n]}} \right) \subseteq \mathcal{X}_{\downarrow \mathbb{Z}_{[1,n]}} \tag{3.47}$$

which is equivalent to a Minkowski sum:

$$\bigoplus_{i=0}^{d_m} A_i \mathcal{X}_{\downarrow \mathbb{Z}_{[in+1,(i+1)n]}} \subseteq \mathcal{X}_{\downarrow \mathbb{Z}_{[1,n]}} = P \tag{3.48}$$

and leads to the D-invariance conditions for the set P

$$\bigoplus_{i=0}^{d_m} A_i P \subseteq P \tag{3.49}$$

thus completing the proof. □

Theorem 3.7. *The system (3.34) admits a family of cyclic D-invariant sets in $\mathbb{R}^n$ if and only if there exists a family of cyclic invariant sets $\{\mathcal{X}_0, \dots, \mathcal{X}_{d_m}\} \subseteq \{\mathbb{R}^n\}^{d_m+1}$ with respect to (3.42) such that*

- *each set $\mathcal{X}_i, i \in \mathbb{Z}_{[0,d_m]}$ admits a balanced ordered factorization with the same partition of indices;*
- *these factorizations share the same family of $d_m + 1$ factors, their order inside the Cartesian product being obtained via a circular shift.*

Proof. Analogous to Theorem 3.6. □

A similar result can be stated for the invariant family of sets with respect to (3.34). In this case the balanced ordered factorizations will be done on a larger family of factors, the circular shift condition being relaxed.

3.5.3 The generalization of the set factorization

From the elements provided in the previous section, it becomes clear that:

- The *extended state space invariance* corresponds to a *minimal* factorization. This is obvious by the fact that, in this particular case, there exists a single factor in the factorization, the set itself.
- Under the constraints imposed by the dimension of the original delay difference equation (3.34), the D-invariance represents the *maximal* regular (ordered) factorization. In order to establish an invariance related result, under the assumption that the state space model (3.34) is minimal, the dimension of the factors in the factorization process is lower bounded by n. Taken into account that the extended state space (3.42) has dimension $(\mathbb{R}^n)^{d_m+1}$, the number of factors cannot be increased above $d_m + 1$.

The regular ordered polyhedral set factorization passes by a block organisation of the set of constraints in the half space description as detailed in Proposition 3.4. A natural way of transforming the structure of a given polyhedral set $\mathcal{X} \in (\mathbb{R}^n)^{d_m+1}$ in view of factorization is based on linear similarity state transformations. The next result summarizes the degrees of freedom in this respect and its proof is a direct consequence of Proposition 3.4.

Proposition 3.5. *Let an extended invariant set* $\mathcal{X} = \left\{X \in (\mathbb{R}^n)^{d_m} | FX \leq w\right\}$ *with respect to the time-delay system* (3.42)*. A regular ordered factorization with dimension-n factors exists if there exists a transformation matrix* $T \in \mathbb{R}^{n(d_m+1)\times n(d_m+1)}$ *such that:*

$$FT^{-1} = \begin{bmatrix} F_0 & 0 & \dots & 0 \\ 0 & F_1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & F_{d_m} \end{bmatrix} \tag{3.50}$$

Corollary 3.2. *Let a delay-difference equation described by* (3.34)*. There exists a D-invariant set for this dynamical system in* $\mathbb{R}^n$ *if the following conditions are fulfiled:*

- *There exists a similarity transformation T such that*

$$\begin{bmatrix} B_0 & \dots & B_{d_m-1} & B_{d_m} \\ I & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & I & 0 \end{bmatrix} = T \begin{bmatrix} A_0 & \dots & A_{d_m-1} & A_{d_m} \\ I & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & I & 0 \end{bmatrix} T^{-1} \tag{3.51}$$

- *There exists an invariant set with respect to*

$$\tilde{X}[t_{k+1}] = \begin{bmatrix} B_0 & \dots & B_{d_m-1} & B_{d_m} \\ I & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & I & 0 \end{bmatrix} \tilde{X}[t_k] \tag{3.52}$$

  which admits a regular ordered factorization.

Sketch of proof: The first condition introduce the similarity transformation in the construction of D-invariant sets all by preserving the dynamical model in the form of a delay difference equation in $\mathbb{R}^n$. Following the results of the Proposition 3.5, such a similarity transformation represents a parametrization of the conditions for the existence of a regular ordered factorization. The second condition of the Corollary recalls the necessary and sufficient conditions in Theorem 3.6.

With linear algebra manipulations, it can be shown that the constraint imposed on the similarity transformation (3.51) is very restrictive[5] and can allow only simple change of coordinates on the original delay difference equation, without a major impact on the regular ordered factorization. However the result has an interesting consequence as it opens the way for factorizations which are in between the *minimal* (extended state space invariant set) and the *maximal* (the D-invariant set). The idea is to find a similarity transformation

$$\begin{bmatrix} B_0 & \dots & B_{r-1} & B_r \\ I & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & I & 0 \end{bmatrix} = T^{-1} \begin{bmatrix} A_0 & \dots & A_{d_m-1} & A_{d_m} \\ I & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & I & 0 \end{bmatrix} T \tag{3.53}$$

which corresponds to a non-minimal state space equation (3.34) as long as $r < d_m$. A simple numerical example will illustrate in the next section the fact that there might exist set-factorizations leading to invariant structures with an intermediate complexity in between $\mathbb{R}^n$ and $(\mathbb{R}^n)^{d_m+1}$.

[5] It practically holds for block diagonal matrices T.

Example 3.1. Consider the delay difference equation:

$$x[t_{k+1}] = x[t_k] - 0.5x[t_{k-2}] \tag{3.54}$$

which does not allow a $D-$invariant set. Its extended state realization:

$$\begin{bmatrix} x_{k+1} \\ x_k \\ x_{k-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -0.5 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x[t_k] \\ x[t_{k-1}] \\ x[t_{k-2}] \end{bmatrix} \tag{3.55}$$

has a strictly stable transition matrix and by consequence allows the construction of invariant sets in $\mathbb{R}^3$. However these invariant sets are not factorisable in $\mathbb{R}$ and following Theorem 3.6 cannot lead to D-invariant sets with respect to (3.54).

Interestingly, there exists a *nonminimal* state delay difference equation equivalent with (3.54):

$$\begin{bmatrix} x_{k+1} \\ x_{k+1} - x_k \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_k \\ x_k - x_{k-1} \end{bmatrix} + \begin{bmatrix} 0 & 0.5 \\ -0.5 & 0.5 \end{bmatrix} \begin{bmatrix} x_{k-1} \\ x_{k-1} - x_{k-2} \end{bmatrix} \tag{3.56}$$

for which a $D-$invariant set $P \subset \mathbb{R}^2$ exists, an example being depicted in Fig. 3.3.

$$P = \left\{ \begin{bmatrix} 0.7071 & -0.7071 \\ 0 & -1.0000 \\ -1.0000 & 0 \\ -0.7071 & 0.7071 \\ -0.4472 & 0.8944 \\ 0.4472 & -0.8944 \\ 0.0000 & 1.0000 \\ 1.0000 & 0.0000 \end{bmatrix} x \leq \begin{bmatrix} 0.2571 \\ 0.1970 \\ 0.3030 \\ 0.2571 \\ 0.2439 \\ 0.2439 \\ 0.1970 \\ 0.3030 \end{bmatrix} \right\}$$

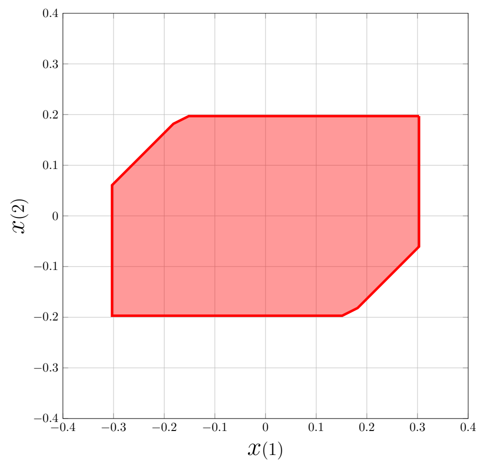


FIGURE 3.3: D-invariant set for the non-minimal state-space representation

This will represent the factor of an invariant set in $\mathbb{R}^4$ with respect to the dynamics in a non-minimal state space:

$$\begin{bmatrix} x[t_{k+1}] \\ x[t_{k+1}] - x[t_k] \\ x[t_k] \\ x[t_k] - x[t_{k-1}] \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 & 0 & 0.5 \\ 0 & 0 & -0.5 & 0.5 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x[t_k] \\ x[t_k] - x[t_{k-1}] \\ x[t_{k-1}] \\ x[t_{k-1}] - x[t_{k-2}] \end{bmatrix} \tag{3.57}$$

3.6 Concluding remarks

This chapter was dedicated to the positive invariance for discrete time-delay systems, where we considered this notion with respect to augmented and initial state-space. If it exists for a given dynamics, positively invariant set defined for the initial state-space is preferable because of its simpler representation. However, it is not certain when such a set might exist. This problem will be addressed in the subsequent chapter. As a contribution with respect to the state of the art, we provided results on the characterization of the mRDI set. In the last section of this chapter a novel insight is provided on alternative solutions for invariance for dDDE. Moreover, it was shown that a set-factorization is the key operation allowing the description of invariant sets in state-spaces of different dimensions. This flexibility opens new perspectives for a better complexity management of the constraints describing the invariant sets. Subsequently, low complexity invariant sets have implications in the complexity of the control design as for example in the MPC framework.

Chapter 4

Delay-difference equations. Stability and positive invariance

THE objective of this chapter is to characterize the existence of positively invariant sets for the linear discrete-time delay-difference equations. The angle considered in this chapter is completely different with respect to the existing approaches in the literature. In order to decrease the conservativeness of the time-domain methods (see Gielen et al. [2012b], Lombardi et al. [2011a]), in this study we use the *frequency-domain framework*. In particular, a strong stability of dDDE, denoted by *robust asymptotic stability*, and its relation with the D-invariance is examined. This notion is referred to as "strong" because it defines stability with respect to all delay realizations. However, characterization of the robust asymptotic stability is not simple mainly because of "incompleteness" of the discrete-time representation. Hence we turn to a more general class of difference equations, the ones that are specified in the continuous-time domain.

Linear continuous-time delay-difference equations (cDDE) are largely treated in the literature, mainly in the context of *neutral functional differential equations* (see e.g. Hale and Lunel [1993]) where they play an important role in the stability analysis. One particularity of the cDDE is their "sensitivity" to (arbitrarily small) delay perturbation (see Michiels et al. [2002]). Hence strong necessary and sufficient stability conditions were proposed (Silkowski [1976]). In this chapter the concept of strong stability is denoted by *delay-independent stability*[1] and it is regarded as the continuous-time counterpart to the robust asymptotic stability. The importance of cDDE is the new insight that this class of system provides in analyzing the robust asymptotic stability and the existence of $D-$invariant sets. The last statement represents the main research path in this chapter.

[1] This notion is also known as *stability in the delays* (see Hale and Lunel [1993], Carvalho [1996])

4.1 Discrete-time delay-difference equations

In this section we consider stability of the linear discrete-time delay-difference equations. Problem of stabilization is not treated here, thus we will be analyzing the following model:

$$x[t_k] = \sum_{i=1}^{m} A_i x[t_{k-d_i}], \tag{4.1}$$

where $x[t_{k-d_i}] \in \mathbb{R}^n$, $A_i \in \mathbb{R}^{n\times n}$ and $d_i \in \mathbb{Z}_{[1,\infty]}$ such that $0 < d_i < d_{i+1},\ \ \forall i \in \mathbb{Z}_{[1,d_m]}$. System (4.1) can be regarded as a closed-loop discrete-time delay difference equation with respect to a linear controller.

The following definitions are used throughout this discussion.

Definition 4.1. *The trivial solution of* (4.1)*,* $x[t_{k-d_i}] = 0,$ *where* $i \in \mathbb{Z}_{[1,m]}$*, is stable if* $\forall \epsilon \in \mathbb{R}_{(0,\infty)}\ \exists \delta(\epsilon) \in \mathbb{R}_{(0,\infty)}$ *such that* $\sup_i \|x[t_{k-d_i}]\|_p \leq \delta$ *implies* $\|x[t_k]\|_p \leq \epsilon$*,* $\forall k \in \mathbb{Z}_{[0,\infty)}$.

Definition 4.2. *The trivial solution of* (4.1)*,* $x[t_{k-d_i}] = 0,$ *where* $i \in \mathbb{Z}_{[1,m]}$*, is attractive if* $\|x[t_k]\|_p \to 0$ *when* $k \to \infty$*, for any* $x[t_{-d_i}] \in \mathbb{R}^n,\ \ i \in \mathbb{Z}_{[1,d_m]}$.

Definition 4.3. *The linear dDDE* (4.1) *is asymptotically stable if its trivial solution is stable and attractive.*

Remark 4.1. Due to linearity, if (4.1) is asymptotically stable then it is globally asymptotically stable.

Delays form a vector $\vec{d} = [d_1 \dots d_m]^T \in (\mathbb{Z}^+)^m$ in the delay-parameter space (notice the red dots on Fig. 4.1). For each $\vec{d}$ we define a *discrete ray*

$$\mathcal{T}_d(\vec{d}) = \{\alpha \vec{d}\ :\ \alpha \in \mathbb{Z}^+\}. \tag{4.2}$$

To estimate asymptotic stability of (4.1), one can augment the state space by rewriting all delayed states as a column vector. Without loss of generality, assume that $\vec{d} = \begin{bmatrix} 1 & 2 & \dots & m \end{bmatrix}^T$. Then, the corresponding augmented state-space representation is written as:

$$X[t_k] = \mathbb{A}(\vec{d})X[t_{k-1}] = \begin{bmatrix} A_1 & \dots & A_{m-1} & A_m \\ I_n & \dots & 0_{n\times n} & 0_{n\times n} \\ \vdots & \ddots & \vdots & \vdots \\ 0_{n\times n} & \dots & I_n & 0_{n\times n} \end{bmatrix} X[t_{k-1}], \tag{4.3}$$

where $X[t_{k-1}] = \begin{bmatrix} x[t_{k-1}]^T & x[t_{k-2}]^T & \dots & x[t_{k-m}]^T \end{bmatrix}^T$. This reformulation has been already vastly used in the previous discussion so it will not be detailed again. We will

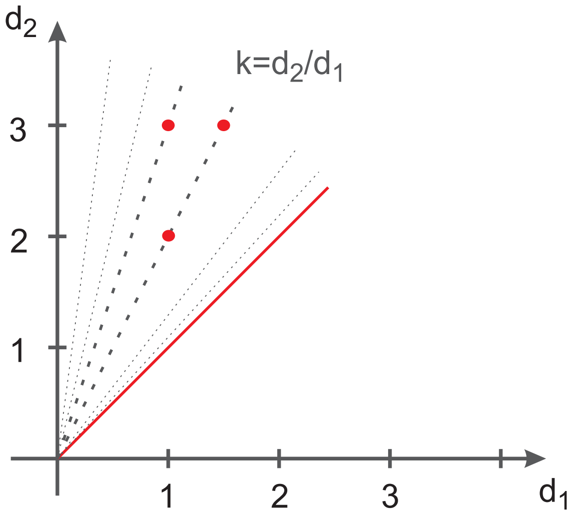


FIGURE 4.1: Delay-parameter space for dDDE when $m = 2$

only stress that, regardless of delay values in (4.1), it is always possible to obtain such a representation, where $\mathbb{A}$ is a function of $\vec{d}$.

The following results are well-known and they are reported without proofs (see e.g. Aström and Wittenmark [1997]).

Lemma 4.1. *The following statements hold:*

i. System (4.1) is asymptotically stable if and only if

$$\det\left(I - \sum_{i=1}^{m} A_i z^{-i}\right) \neq 0, \quad \forall z \in ext(\mathbb{D}) \cup \partial\mathbb{D}; \tag{4.4}$$

ii. System (4.3) is asymptotically stable if and only if the spectral radius of the matrix $\mathbb{A}(\vec{d})$ satisfies

$$\rho(\mathbb{A}(\vec{d})) < 1. \tag{4.5}$$

The following theorem links the two stability conditions from the previous lemma.

Theorem 4.1. *The following statements are equivalent:*

i. The dDDE (4.1) is asymptotically stable;

ii. The linear equation (4.3) is asymptotically stable.

Proof. Consider the characteristic equation of (4.3)

$$\det\left(I_{nm} - z^{-1}\mathbb{A}(\vec{d})\right) = \det\left(\left[\begin{array}{c|ccc} I_n - A_1 z^{-1} & \dots & -A_{m-1}z^{-1} & -A_m z^{-1} \\ \hline -I_n z^{-1} & \dots & 0_{n\times n} & 0_{n\times n} \\ \vdots & & \vdots & \vdots \\ 0_{n\times n} & \dots & -I_n z^{-1} & I_n \end{array}\right]\right). \quad (4.6)$$

Define the following blocks for the previous conformably partitioned matrix (see Meyer [2000]):

$$P_{11} = I_n - A_1 z^{-1}, \quad P_{12} = \begin{bmatrix} -A_2^T z^{-1} \\ \vdots \\ -A_m^T z^{-1} \end{bmatrix}^T,$$

$$P_{21} = \begin{bmatrix} -I_n z^{-1} \\ \vdots \\ 0_{n\times n} \end{bmatrix}, P_{22} = \begin{bmatrix} I_n & \dots & 0_{n\times n} & 0_{n\times n} \\ \vdots & & \vdots & \vdots \\ 0_{n\times n} & \dots & -I_n z^{-1} & I_n \end{bmatrix}.$$

Block P_{22} is invertible for any $z^{-1} \neq 0$. Therefore, the characteristic equation (4.6) can be rewritten as:

$$\det(S) = 0,$$

where $S = P_{11} - P_{12}P_{22}^{-1}P_{21}$ is the Schur complement where

$$P_{22}^{-1} = \begin{bmatrix} I_n & 0_{n\times n} & \dots & 0_{n\times n} \\ z^{-1}I_n & I_n & \dots & 0_{n\times n} \\ \vdots & \vdots & & \vdots \\ z^{-m+1}I_n & z^{-m+2}I_n & \dots & I_n \end{bmatrix}.$$

After multiplying matrices in the Schur complement expression, (4.6) is transformed into

$$\det\left(I_n - A_1 z^{-1} - \ldots - A_m z^{-m}\right) = \det\left(I_n - \sum_{i=1}^{m} A_i z^{-i}\right). \quad (4.7)$$

Assume that the statement $i.$ of the theorem holds, i.e., $\det\left(I_n - \sum_{i=1}^{m} A_i z^{-i}\right) \neq 0 \;\; \forall z \in ext(\mathbb{D}) \cup \partial\mathbb{D}$. Then, according to (4.7), it follows that $\det\left(I_{nm} - z^{-1}\mathbb{A}(\vec{d})\right) \neq 0 \;\; \forall z \in ext(\mathbb{D}) \cup \partial\mathbb{D}$. The same holds for the $ii.$ to $i.$ implication of the proof. $\square$

Denote by z^* all eigenvalues of $\mathbb{A}(\vec{d})$. In the following lines we consider stability of dDDEs by regarding the delay parameter space. First, assume that delays in (4.1) vary in such a way that delay vector always stays on the same ray, i.e., $\vec{d} \in \mathcal{T}_d(\vec{d})$. Stability of (4.1) with respect to this delay variation is characterized by the following spectrum:

$$\sigma(A(\alpha\vec{d})) = \left\{ z_\alpha^* \ : \ \det\left(I_{nm} - \sum_{i=1}^{m} A_i z_\alpha^* = 0 \right), \ \alpha \in \mathbb{Z}^+ \right\}. \tag{4.8}$$

One can notice that $z^* = z_\alpha^*$. Hence, if the null solution of the dDDE (4.1) is asymptotically stable for a vector $\vec{d}$, it will remain asymptotically stable $\forall \vec{d} \in \mathcal{T}_d(\vec{d})$. We denote a ray with this property as *stable*.

From the positive invariance point of view, we are more interested in stability that concerns general delay variation (variation over all rays). According to Verriest and Ivanov [1995], we introduce the following definition of the robust (with respect to delay uncertainty) stability.

Definition 4.4. *The dDDE (4.1) is robustly asymptotically stable if its null solution is asymptotically stable* $\forall \vec{d} \in (\mathbb{Z}^+)^m$.

This notion was considered by several authors. For instance, in Verriest and Ivanov [1995] several sufficient conditions were proposed by using Riccati-type equations. Also, in Lombardi et al. [2011b] we can find the following simple sufficient condition which was initially proposed by Kharitonov [1991] for the continuous-time delay-difference equations.

$$\sum_{i=1}^{m} \|A_i\| < 1. \tag{4.9}$$

Note that (4.9) is a necessary and sufficient robust stability condition for the scalar case. On the other side, in Gielen et al. [2012b] authors provided a necessary condition for the robust stability via asymptotic stability of a family of linear systems. This result is outlined in the following theorem.

Theorem 4.2 (Gielen et al. [2012b])**.** *Let* $\mathbb{S} = \{-1, 0, 1\}$ *and* $\Delta = \begin{bmatrix} \delta(1) & \dots & \delta(m) \end{bmatrix}$. *Consider the following family of systems:*

$$z[t_{k+1}] = \sum_{i=1}^{m} \delta(i) A_i z[t_k], \quad \Delta \in \mathbb{S}^m. \tag{4.10}$$

Assume that (4.1) is robustly assymptotically stable. Then (4.10) is asymptotically stable $\forall \Delta \in \mathbb{S}^m$.

Proof. See Gielen et al. [2012b]. □

Several other necessary conditions, such as $\rho(\mathbb{A}(\vec{d})) < 1$ and $\rho\left(\sum_{i=1}^{m} A_i\right) < 1$, can be found in Lombardi et al. [2011b]. However none of the presented conditions is necessary and sufficient, and for each of them it is relatively simple to construct a counter example, as illustrated at the end of this chapter.

4.1.1 Stability results in term of Lyapunov functions

For the sake of completeness, we also provide a Lyapunov-type stability conditions for dDDE.

Theorem 4.3. *Let α_1, $\alpha_2 \in \mathcal{K}_\infty$ and $\epsilon \in \mathbb{R}_{[0,1)}$. The following statements are equivalent:*

i. For all $X[t_k] \in (\mathbb{R}^n)^m$ there exists a Lyapunov-Krasovskii function $V : (\mathbb{R}^n)^m \to \mathbb{R}_{[0,\infty)}$ with respect to the equation (4.3) such that:

$$\alpha_1(\|X[t_k]\|) \leq V(X[t_k]) \leq \alpha_2(\|X[t_k]\|), \quad V(X[t_i]) \leq \epsilon V(X[t_{k-1}]);$$

ii. The linear difference equation (4.3) is globally asymptotically stable;

Namely, for infinite-dimensional systems Lyapunov-Krasovskii functional uses trajectory segments and is strictly decreasing along all solutions for a given time-delay system (see Niculescu [2001]). Analogously, in the discrete-time case trajectory segments are represented by finite-dimensional state sequences.

Theorem 4.4. *Consider the system represented by (4.1). Let $V : (\mathbb{R}^n)^m \to \mathbb{R}_{[0,\infty)}$, α_1, $\alpha_2 \in \mathcal{K}_\infty$ and $\epsilon \in \mathbb{R}_{[0,1)}$. Let*

$$\bar{V}(X[t_k]) := \max_{i \in \mathbb{Z}_{[1,m]}} \{V(x[t_{k-i}])\}. \tag{4.11}$$

If for all $x \in \mathbb{R}^n$, $k \in \mathbb{Z}_{[0,\infty)}$ and $X \in (\mathbb{R}^n)^m$

$$\begin{aligned} &\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \\ &V(x[t_k]) - \epsilon \bar{V}\,(X[t_{k-1}) \leq 0, \end{aligned} \tag{4.12}$$

then, the system is asymptotically stable.

Proof. See Gielen [2013]. □

Stability considered in Theorem 4.4 represents a stronger notion than stability based on Lyapunov-Krasovksii approach from Theorem 4.3 and it is referred to as *Lyapunov-Razumikhin*-based stability condition (see Lombardi et al. [2011a], Gielen et al. [2012b]). Regarding these notions with respect to positive invariance, it is clear that for a given dynamics, Lyapunov-Krasovskii stability condition is necessary and sufficient for the existence of positively invariant sets in the augmented state-space. This observation is rather obvious since the augmented state-space representation is a LTI system. Analogously, in Gielen et al. [2012a] it was shown that Theorem 4.4 provides necessary and sufficient condition for the existence of $D-$invariant sets. Even though Lyapunov-Razumikhin approach formally solves the problem of the existence for D-invariant sets, it still does not provide a constructive result which would state whether such a function exists for a system of interest. Therefore, defining a condition for the existence of $D-$invariant sets would also mean solving the problem of the existence for Lyapunov-Razumikhin functions for linear dDDE. In what follows we offer an answer to this problem for the linear delay-difference equations with two delay parameters.

4.2 Continuous-time delay-difference equations

When the robust asymptotic stability is examined, one may perceive that the discrete-time domain is somehow "incomplete" regarding delay parameters (see Fig. 4.1). Namely, it can be noticed that for small perturbation of a stable ray, dDDE (4.1) can become unstable. This trend is even more noticeable for larger α in (4.8). For instance, let us consider the following example (see Michiels and Niculescu [2013]).

Example 4.1. Given the stable dDDE

$$x[t_k] = 3/4x[t_{k-d_1}] - 1/2x[t_{k-d_2}]. \tag{4.13}$$

For $d_1 = 1$ and $d_2 = 2$ all characteristic roots are inside the unit disk, i.e., $|z_{max}| = 0.7071$, where

$$|z_{max}| = \max_i\{|z_i| \;:\; \det(1 - \frac{3}{4}z^{-d_1} + \frac{1}{2}z^{-d_2})\}.$$

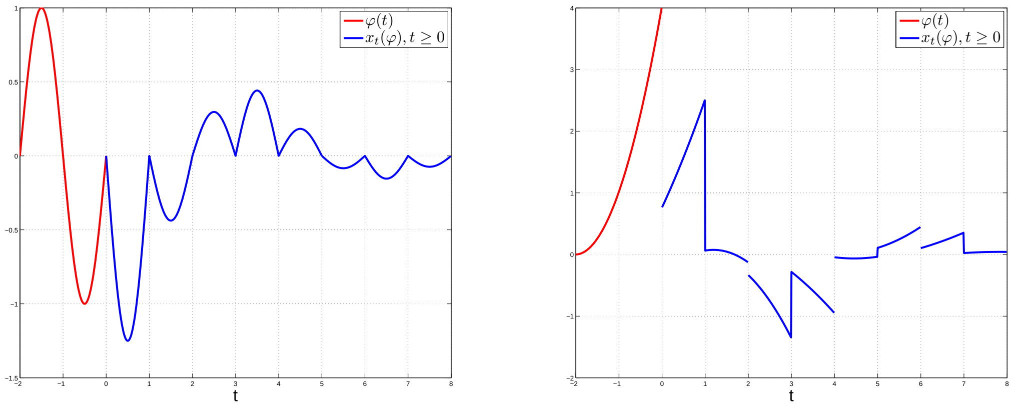
For $\alpha = 10$, i.e., for $d_1 = 10$ and $d_2 = 20$, and according to the discussion which follows (4.8), the system is stable (delays from the same ray) but with the smaller stability margin. Indeed, $|z_{max}| = 0.9659$. The same holds also for $d_1 = 11$ and $d_2 = 22$, with $|z_{max}| = 0.9690$. However, a small perturbation of the direction of this ray leads to instability. For instance, for $d_1 = 10$ and $d_2 = 21$ we obtain $|z_{max}| = 1.0159$.

The previous example displayed *sensitivity* of the stability property for dDDE to a small delay variation. It is clear that by using discrete-time representation one cannot properly handle this phenomena. For this reason we regard the general class of difference equations, the continuous-time delay-difference equations (cDDEs).

Consider the cDDE:

$$x(t) = \sum_{i=1}^{m} A_i x(t - r_i), \tag{4.14}$$

where $t \in \mathbb{R}_+$, $A_i \in \mathbb{R}^{n\times n}$, $r_i \in \mathbb{R}^+$, $i \in \mathbb{Z}_{[1,m]}$. It is clear that (4.1) is obtained as a special case of (4.14) by restricting the continuous time variable t to the discrete-time



(A) $\varphi \in \mathcal{C}(\mathbb{R}_{[-r_m,0]}, \mathbb{R}^n)$ (B) $\varphi \in \mathcal{C}(\mathbb{R}_{[-r_m,0)}, \mathbb{R}^n)$

FIGURE 4.2: Forward solution of cDDE

sequences t_k. For every initial condition $\varphi \in \mathcal{C}_D(\mathbb{R}_{[-r_m,0]}, \mathbb{R}^n)$, a forward solution of (4.14) on $\mathbb{R}_{[-r_m,0]}$ is uniquely defined. The space of initial functions is defined by:

$$\mathcal{C}_D = \{\varphi \in \mathcal{C}(\mathbb{R}_{[-r_m,0]}, \mathbb{R}^n) \ : \ \varphi(0) = \sum_{i=1}^{m} A_i \varphi(r_i)\}. \tag{4.15}$$

Notice that the algebraic constraint in (4.15) provides continuity of the solution. As an example of solution with respect to continuous and discontinuous initial function see Fig. 4.2a and Fig. 4.2b, respectively. In general, appropriate functional space for the initial value problem is referred to as a *phase space* (see Carvalho [1996]) which is a Banach space invariant with respect to the operator given by the right-hand-side of (4.14).

Denote by $x_t(\varphi) = \{x(\theta) \ : \ x(\theta) = \sum_{i=1}^{m} A_i x(\theta - r_i), \ \theta \in \mathbb{R}_{[0,t]}\}$.

Definition 4.5. *Let $\varphi \in \mathcal{C}_D$. The trivial solution of the cDDE (4.14) is stable if $\forall \epsilon > 0$ $\exists \delta > 0$ such that whenever $\|\varphi\|_p < \delta$, $\|x_t(\varphi)\|_p < \epsilon$, $\forall t \in \mathbb{R}_+$.*

Definition 4.6. *Let $\varphi \in \mathcal{C}_D$. The trivial solution of the cDDE (4.14) is attractive if $\|x_t(\varphi)\|_p \to 0$ as $t \to \infty$.*

Definition 4.7. *The linear cDDE (4.14) is asymptotically stable if its trivial solution is stable and attractive.*

As for the discrete-time case, the *global* feature of the previous definitions is inherent due to the linearity.

Arranging delays into a vector we define $\vec{r} = [r_1 \dots r_m]^T \in (\mathbb{R}^+)^m$, where $r_i < r_{i+1}$. For each $\vec{r}$ it is possible to define a *ray*

$$\mathcal{T}_c(\vec{r}) = \{\beta\vec{r} \ : \ \beta \in \mathbb{R}^+\}. \tag{4.16}$$

Stability of cDDE (4.14) is determined by the roots of exponential polynomial of the form

$$P(\lambda) = 1 - \sum_{i=1}^{m} A_i e^{-\lambda r_i}. \tag{4.17}$$

Namely, denote by

$$c(\vec{r}) = sup\,\{\Re(\lambda) \ : \ P(\lambda) = 0\} \tag{4.18}$$

the spectral abcissa. Stability of (4.14) is determined according to the following theorem.

Theorem 4.5. *Delay-difference equation (4.14) is stable if and only if $c < 0$.*

Proof. See Hale and Lunel [1993]. □

Likewise the discrete-time case, if the cDDE (4.14) is asymptotically stable for a vector $\vec{r} \in \mathcal{T}_c(\vec{r})$, it also remains asymptotically stable $\forall \vec{r} \in \mathcal{T}_c(\vec{r})$. Such a ray is denoted as stable. However, stability of a ray is a "sensitive" property for cDDEs. As for the discrete-time case, arbitrarily small perturbations in the delay vector lead to the occurrence of sequences of characteristic roots whose imaginary parts grow unbounded, yet whose real parts have a finite limit. With increasing the modulus of these roots their "sensitivity" to delay perturbation also grows and it can become unboundedly large (see Michiels et al. [2002]). This phenomenon is also known as *delay interference* and it may regularly appear in systems described by delay-differential equations (see Louisell [1995], Michiels and Niculescu [2007]). Similarly to the discrete-time case, here we also define a strong stability notion (with respect to delay variation) for cDDE (notice the region in grey depicted on Fig. 4.3).

Definition 4.8. *The cDDE (4.14) is delay-independently stable if its null solution is asymptotically stable $\forall \vec{r} \in (\mathbb{R}^+)^m$.*

Remark 4.2. Even though the robust asymptotic stability is sometimes referred to as delay-independent stability (see e.g. Boukas [2006]), in this study we a make distinction between them and we relate them with dDDE and cDDE respectively. The reason for this is that it is not clear whether the robust asymptotic stability approaches the delay-independent stability when $r_m \to \infty$. This theoretical problem is still not solved.

It was shown in Avellar and Hale [1980] that the spectral abscissa is not continuous with respect to delays. Therefore, when the delay-independent stability is considered, we are

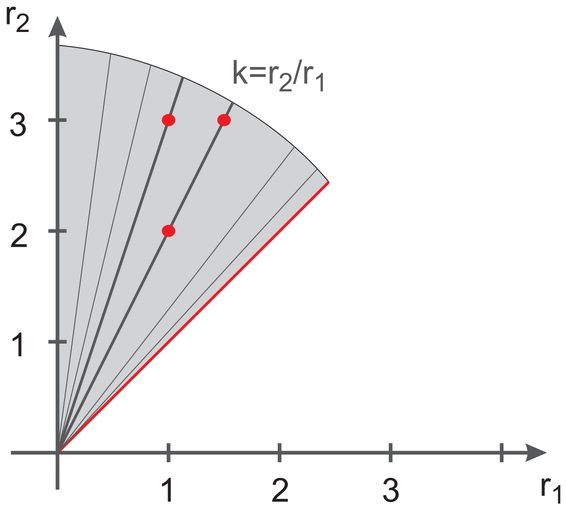


FIGURE 4.3: Delay-parameter space for cDDE when $m = 2$

interested in the robust spectral abscissa (see Michiels and Niculescu [2013]) which is defined as:

$$\bar{c}(\vec{r}) = \lim_{\epsilon \to 0^+} sup \left\{ c(\vec{r} + \delta\vec{r}) \ : \ \|\delta\vec{r}\|_2 \leq \epsilon \right\}. \tag{4.19}$$

The robust spectral abscissa is a continuous function with respect to delays. This property will be used later in proving the necessary and sufficient condition for delay-independent stability.

By denoting $\vec{\theta} = [\theta_1 \dots \theta_m]^T$, the following classical result provides the necessary and sufficient condition for the delay-independent stability (see Silkowski [1976]).

Theorem 4.6. *The following statements are equivalent:*

i. The equation (4.14) *is delay-independently stable;*

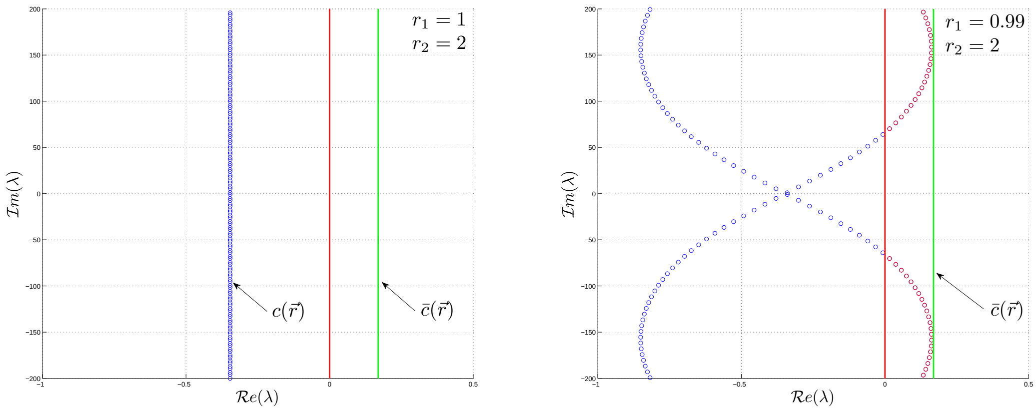
ii. The robust spectral abscissa satisfies $\bar{c}(\vec{r}) < 0$*;*

iii. $\sup \left\{ \rho\Big(\sum_{i=1}^{m} A_i e^{j\theta_i} \Big) \ : \ \vec{\theta} \in [0, 2\pi]^m \right\} < 1.$

Example 4.1 illustrated how delay variation affects the stability of dDDE. In the following example we show how delay variation may affect the cDDE. For this purpose we use the same numerical values as in Example 4.1, but we focus on the continuous-time domain.

Example 4.2. Given the stable dDDE

$$x(t) = 3/4x(t - r_1) - 1/2x(t - r_2). \tag{4.20}$$



(A) $r_1 = 1, \; r_2 = 2$

(B) $r_1 = 0.99, \; r_2 = 2$

FIGURE 4.4: Characteristic roots

For $r_1 = 1$ and $r_2 = 2$ all characteristic roots are in the left complex half space and the system is stable (see Fig. 5.9a). Notice that the poles for the discrete-time case and the continuous-time case are related via transformation $e^{\lambda} = z$. Hence for each pole in the $z-$complex plane there exists an infinite number of poles in the $\lambda-$complex plane with the same real part and arguments with periodicity of 2π.
On the other side, for $r_1 = 0.99$ and $r_2 = 2$ the spectrum of (4.20) is shown in Fig. 5.9b. Notice that the spectral abscissa is not continuous with respect to the delay variation while the robust spectral abscissa is.

If the components of $\vec{r}$ are rationally dependent, then there always exists an integer $p < m$, a vector with rationally independent components $\vec{s} = [s_1 \dots s_p]^T \in \mathbb{R}^p$ and a full column rank matrix $H \in \mathbb{Z}^{m \times p}$ such that $\vec{r} = H\vec{s}$. However, if $p = 1$, then delays $r_1, \dots, r_m$ are called *commensurate* i.e. there exists $h \in \mathbb{R}^+$ and $\vec{d} \in (\mathbb{Z}^+)^m$ such that $\vec{r} = h\vec{d}$ (see Hardy and Wright [1979]). For the vector $\vec{r}$ of commensurate delays, the cDDE (4.14) can be rewritten as

$$x(t) = \sum_{i=1}^{m} A_i x(t - d_i h). \tag{4.21}$$

The last equation represents the continuous equivalent of the dDDE (4.1). This fact will be used later when we state the robust stability results.

The following Lemma, originally proposed for two-dimensional systems (see Fu et al. [2006] and Huang [1972]), provides an efficient numerical method to verify the statement *iii.* from the Theorem 4.6 when $m = 2$. It is worth mentioning that the general extension of this result $(m > 2)$ is an open problem.

Theorem 4.7. *Consider the cDDE (4.14) with $m = 2$. Assume that there exists $\vec{r} \in (\mathbb{R}^+)^2$ for which the system (4.14) is asymptotically stable. Then, the following statements are equivalent:*

i. For $\theta_i \in [0, 2\pi]$, $i = 1, 2$ the following condition holds:

$$\rho\Big(A_1 e^{j\theta_1} + A_2 e^{j\theta_2}\Big) < 1; \tag{4.22}$$

ii. $\sigma(U, V) \cap \partial\mathbb{D} = \emptyset$, where

$$U = \begin{bmatrix} 0 & I \\ -B_0 & -B_1 \end{bmatrix}, \quad V = \begin{bmatrix} I & 0 \\ 0 & B_2 \end{bmatrix}, \tag{4.23}$$

$B_0 = A_1 \otimes A_2^T$, $B_1 = A_1 \otimes A_1^T + A_2 \otimes A_2^T - I$, $B_2 = A_2 \otimes A_1^T$.

Proof. Since there exists an $\vec{r} \in \mathbb{R}^2_{(0,\infty)}$ for which the system (4.14) is asymptotically stable and continuity of $\bar{c}(\vec{r})$ (see (4.19)) holds, delay-independent stability can be confirmed if there are no poles crossing the imaginary axis (see Theorem 4.6). The condition (iii.) from this theorem can be rewritten as

$$\rho\Big(A_1 + A_2 e^{j\theta_2} e^{-j\theta_1}\Big) < 1, \text{ for all } \theta_i \in \mathbb{R}_{[0,2\pi]}. \tag{4.24}$$

Using standard properties of the Kronecker product (see Graham [1981]), condition (4.24) is equivalent to

$$\sigma\Big((A_1 + A_2 z_1) \otimes (A_1 + A_2 z_1)^*\Big) \neq 1 \text{ for all } |z_1| = 1, \tag{4.25}$$

where $z_1 = e^{j\theta_2} e^{-j\theta_1}$ is a complex variable on the unit circle.

Expression (4.25) is equivalent to

$$\det\Big(I - (A_1 + A_2 z_1) \otimes (A_1 + A_2 z_1)^*\Big) \neq 0 \text{ for all } |z_1| = 1. \tag{4.26}$$

For all $|z_1| = 1$, (4.26) can be rewritten as

$$\begin{aligned} &\det\Big(I - A_1 \otimes A_1^T - A_2 \otimes A_2^T - z_1^{-1}(A_1 \otimes A_2^T) - z_1(A_2 \otimes A_1^T)\Big) \\ &= \det\Big(-B_1 - z_1^{-1} B_0 - z_1 B_2\Big) \\ &= \det\Big(z_1 B_1 + B_0 + z_1^2 B_2\Big) \neq 0 \quad \text{for all } \ |z_1| = 1. \end{aligned} \tag{4.27}$$

According to the Schur determinant formula, the previous statement can be rewritten as

$$\det\Big(zV - U\Big) \neq 0 \text{ for all } |z_1| = 1, \tag{4.28}$$

which is, by definition, equivalent to the generalized eigenvalues of the matrices U and V:

$$\sigma(U, V) \cap \mathbb{D} = \emptyset.$$

□

The proposed algebraic condition will play the major role in the later examination of the set-invariance property for the delay-difference equations.

4.3 Computational conditions for set invariance existence

Even though the system (4.21) is in the continuous-time domain, it is finite-dimensional and, from the stability point of view, it is equivalent to the dDDE (4.1). This was shown in the previous example where, by using the substitution $z = e^{sh}$, it was demonstrated that for each pole $z \in \mathbb{D}$ of the dDDE, there exists an infinite number of poles $s \in \mathbb{C}^-$ of the cDDE with the same real part and the argument with periodicity of 2π. Let us introduce now $Q(s, \vec{r}) = \det\big(I - \sum_{i=1}^{m} A_i e^{-sr_i}\big)$, $\sigma(\vec{r}) = \{s : Q(s, \vec{r}) = 0\}$ and the sets

$$W = \bigcup_{\vec{r} \in (\mathbb{R}^+)^m} \sigma(\vec{r}) \tag{4.29}$$

$$V_h = \bigcup_{\vec{d} \in (\mathbb{Z}^+)^m} \sigma(h\vec{d}), \text{ for some } h \in \mathbb{R}^+ \tag{4.30}$$

Proposition 4.1. *For all positive $h \in \mathbb{R}^+$, $V_h \subset W$. Furthermore, $\bigcup_{h \in \mathbb{R}^+} V_h \subset W$.*

Proof. Let $\mathcal{R} = \{\vec{r} \ : \ \vec{r} \in (\mathbb{R}^+)^m\}$ and $\mathcal{R}_c = \{\vec{r} \ : \ \vec{r} \in (\mathbb{R}^+)^m, \ \vec{r} = h\vec{d}, \ h \in \mathbb{R}^+, \ \vec{d} \in (\mathbb{Z}^+)^m\}$. The proof of the proposition arises immediately from the fact that $\mathcal{R}_c \subset \mathcal{R}$. □

Corollary 4.1. *If* $\sup\Big\{\rho\Big(\sum_{i=1}^{m} A_i e^{j\theta_i}\Big) : \vec{\theta} \in [0, 2\pi]^m\Big\} < 1$ *then* $\sigma(A(\vec{d})) \in \mathbb{D}$, $\forall \vec{d} \in (\mathbb{Z}^+)^m$.

Remark 4.3. The Corollary 4.1 provides the sufficient condition for the robust asymptotic stability of the dDDE (4.1). However, the following remains unclear: if a necessary and sufficient condition for the robust asymptotic stability existed, would its fulfillment be sufficient as well to guarantee the delay-independent stability? In what follows, we examine how the existence of the D-invariant sets can provide some new perspectives to this question.

Subsequently, we address the correlation between the robust asymptotic stability and the existence of the D-contractive sets via frequency-domain approach. The link between the discrete-time (4.1) and the continuous-time dynamics (4.14) becomes natural if the D-invariance notion is defined with respect to the continuous time dynamics. This definition is introduced next through Minkowski sum construction and keeps the coherence with the discrete version in Definition 3.26 (for $\omega = 0$).

Definition 4.9. *Regarding the system* (4.14) *and* $0 \leq \epsilon \leq 1$*, the set* $\mathcal{X} \subseteq \mathbb{R}^n$*, containing the origin in its strict interior, is said to be* $D-$*invariant if* $\bigoplus_{i=1}^{m} A_i\mathcal{X} \subset \epsilon\mathcal{X}$.

With the remarks and definition above, an important property of the $D-$invariant sets is characterized by the following proposition.

Proposition 4.2 (Stanković et al. [2013])**.** *The set* $\mathcal{X}$ *is* $D-$*invariant with respect to the dDDE* (4.1) *if and only if it is D-invariant with respect to the cDDE* (4.14)*.*

Proof. Regarding Definition 4.9, one can notice that the existence of a D-invariant set is not related to the continuous or discrete-time domain of the delay-difference equations but only to the linear mappings that define the dynamics in (4.1) and (4.14), which are essentially the same. □

As a consequence of the Proposition 4.2, instead of using "incomplete" discrete-time representation (4.1) for defining the conditions for the existence of the $D-$contractive sets, we may equally employ general continuous-time model (4.14) and the related frequency-domain techniques. Before stating the main constructive result, a useful intermediary statement is needed.

Lemma 4.2 (Stanković et al. [2013])**.** *If the cDDE* (4.14) *admits a* $D-$*contractive set* $\mathcal{X}$ *including the origin, then for any initial condition given by a continuous function* $\varphi : [-r_m, 0) \to \mathcal{X}$*, the system's trajectories stay inside* $\mathcal{X}$ *and converge to the origin.*

Proof. The proof is based on an appropriate inductive argument. From the initial condition constraint, it follows that for all $t \in [-r_m, 0)$ the system trajectory $x(t) \in \mathcal{X}$. Next, from Definition 4.9, the same property holds on $[0, r_m)$. Furthermore, for a contraction factor $\epsilon \in \mathbb{R}_{[0,1)}$ and due to the fact that $0 \in \mathcal{X}$ we have a strict inclusion $\epsilon\mathcal{X} \subset \mathcal{X}$ leading to $x(t) \in \epsilon\mathcal{X}, \forall t \in [0, r_m)$. The same argument can be used on $[r_m, 2r_m)$ with the remark that the trajectory $x(t) \in \epsilon^2\mathcal{X}, \forall t \in [r_m, 2r_m)$.
Suppose now that, for some $k \geq 2$, the trajectory satisfies $x(t) \in \epsilon^k\mathcal{X}, \forall t \in [(k-1)r_m, kr_m)$. Then similar to the previous remark $x(t) \in \epsilon^{k+1}\mathcal{X}, \forall t \in [kr_m, (k+1)r_m)$.

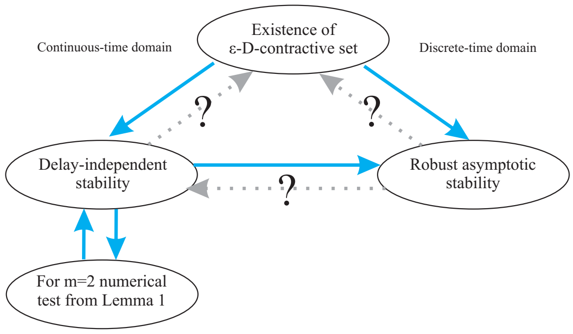


FIGURE 4.5: Schematic overview of the presented results.

This final property completes the proof since when $k \to \infty$ we have $\epsilon^k \to 0$ and subsequently the trajectory approaching the origin stays inside $\mathcal{X}$ and converges asymptotically to the origin. □

The D-contractivity is related to the matrices A_i and it is *independent* on the delay values in (4.14). Lemma 4.2 establishes a connection between the asymptotic stability and the $D-$contractivity of a set with respect to the dynamics in (4.14) and furthermore this result does not rely on the delay values.

Corollary 4.2. *If the cDDE* (4.14) *admits a $D-$contractive set including the origin, then the system is delay-independently stable.*

Theorem 4.8. *The system*

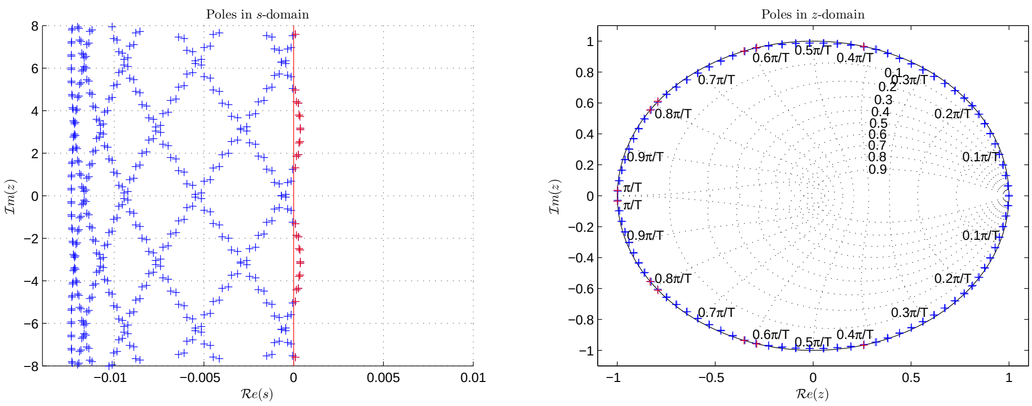
$$x[k] = \sum_{i=1}^{2} A_i x[k - d_i] \tag{4.31}$$

admits the D-contractive sets only if $\sigma(U, V) \cap \partial\mathbb{D} = \emptyset$.

Proof. Since the Theorem 4.7 states the necessary and sufficient condition for the delay-independent stability of the cDDE (4.14) with $m = 2$, according to the Corollary 4.8, it is also necessary to guarantee the existence of the D-contractive sets for the cDDE. Finally, the equivalence between the continuous-time and the discrete-time dynamics is resumed by the Proposition 4.2. □

Remark 4.4. Theorem 4.8 provides the necessary condition for the existence of the D-contractive sets. Furthermore, from the author's experience, a counterexample that shows that this theorem is necessary but not sufficient has not been found yet.

For a schematic overview of the correlation between the notions presented in this note we refer to the Fig. 4.5.



(A) Part of the spectrum of (4.32) for $r_1 = 1$ and $r_2 = 9.1$.

(B) Spectrum of the corresponding dDDE with $d_1 = 10$ and $d_2 = 91$, while $h = 0.1$

FIGURE 4.6: Spectrum for cDDE and dDDE

For a given scalar discrete-time delay-difference equation, a $D-$invariant set can be regarded as an ∞-norm sphere of the dimension that depends on delay value. Therefore, for the scalar dDDE $x[k] = \sum_{i=1}^{m} a_i x[k - d_i]$ the necessary and sufficient condition for the robust asymptotic stability and for the existence of the $D-$invariant sets is $\sum_{i=1}^{m} |a_i| < 1$. This arises from the fact that $\rho(A(\vec{d})) \leq \|A(\vec{d})\|_\infty = 1 \ \forall \vec{d} \in (\mathbb{Z}^+)^m$.

Example 4.3. Let us consider the following asymptotically stable cDDE

$$x(t) = \begin{bmatrix} 0.5 & 0.16 \\ -0.21 & 0.45 \end{bmatrix} x(t-1) + \begin{bmatrix} 0.51 & -0.01 \\ 0.02 & 0.51 \end{bmatrix} x(t-9). \tag{4.32}$$

The corresponding discrete-time representation (4.1), obtained for $h = 1$, $d_1 = 1$ and $d_2 = 9$, is also asymptotically stable i.e. $\rho(A(\vec{d})) = 0.9951$. The necessary condition for the existence of the $D-$contractive sets proposed by Gielen et al. [2012b] (see Theorem 4.2) is fulfilled i.e. $\rho(A_1) < 1$, $\rho(A_2) < 1$, $\rho(A_1 + A_2) < 1$ and $\rho(-A_1 + A_2) < 1$. However, the necessary condition from the Theorem 4.8 does not hold. Indeed, one can verify that the system becomes unstable for the perturbed delay vector $r_1 = 1$ and $r_2 = 9.1$ (spectra of the cDDE and the dDDE are depicted in Fig. 4.6a and Fig. 4.6b, respectively). Consequently, there is no $D-$contractive (neither the D-invariant) set for this example, thus confirming that the prior necessary condition is more restrictive than the Theorem 4.8.

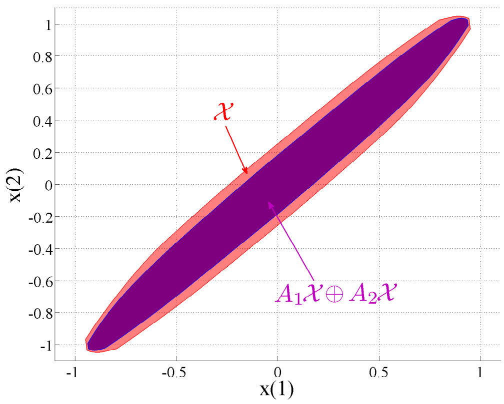


FIGURE 4.7: D-contractive set for the dDDE (4.33).

According to Theorem 4.8, the system

$$x[k] = \begin{bmatrix} 0.35 & 0.13 \\ 0.51 & -0.01 \end{bmatrix} x[k-d_1] + \begin{bmatrix} 0.51 & -0.01 \\ 0.03 & 0.51 \end{bmatrix} x[k-d_2], \tag{4.33}$$

has a cDDE counterpart which is delay-independently stable and implies the robust asymptotic stability of (4.33). Such robust asymptotic stability of the dDDE represents a necessary condition for the existence of a D-contractive set with respect to (4.33). In fact, a D-contractive set exists as shown in Fig. 4.7. Finally, it is interesting to remark that, for this example, the existing sufficient condition $\|A_1\|_p + \|A_2\|_p < 1$ (see Lombardi et al. [2011b]) does not hold.

4.4 Conclusion

In this chapter we examined conditions for the existence of the $D-$contractive sets via strong stability notions for the cDDE and the dDDE. A computationally efficient numerical test, also known from the theory of two-dimensional systems, was therefore employed. We showed that, beside their practical application, $D-$invariant sets can also bring a new insight on the correlation between the robust asymptotic and the delay-independent stability.

Chapter 5

Sensor-to-controller delays. Detection and control design

Knowing the value of an induced delay in communication between sensor and controller, along with sufficiently accurate model of the process, allows us to design a control action which is capable to compensate for such a delay. A classical approach employs a *model-based controller*, which performs estimation (prediction) of a state vector based on available outdated measurements and previous control inputs. However, if the process is not deterministic (it is subject to additive disturbance or parametric uncertainties), this approach may provide poor performance, or even lead to unstable behavior. On the other hand, a lag that might appear in the communication between controllers and actuators has completely different nature. Unless it can be estimated in advance and taken into account in control design, such a delay cannot be compensated by means of networked control. In this situation, one must rely on the robust (with respect to delay parameter) stability analysis and the robust control design (see Chapter 2).

This chapter examines a control design for networked control systems with single sensor-to-controller communication channel. Random and time-varying delays, which can occur during data transmission through this channel, are regarded as faults. Furthermore, we assume that the process is affected by a bounded additive disturbance.
In order to provide information on smaller[1] delays, we design a set-based *delay detection mechanism*, which in general can be regarded as a FDI. An advantage of the proposed mechanism is its simple and numerically low cost implementation because it utilizes set-membership testing to discern a 'healthy' from a 'delayed' information. In order to avoid intersection between regions with 'healthy' and 'delayed' sensor's measurements i.e. to guarantee unique delay detection, a reference governor is designed using *model-based receding horizon optimization* framework. Once information provided by the feedback

[1]The term 'smaller' will be precisely quantified in Section 5.3.

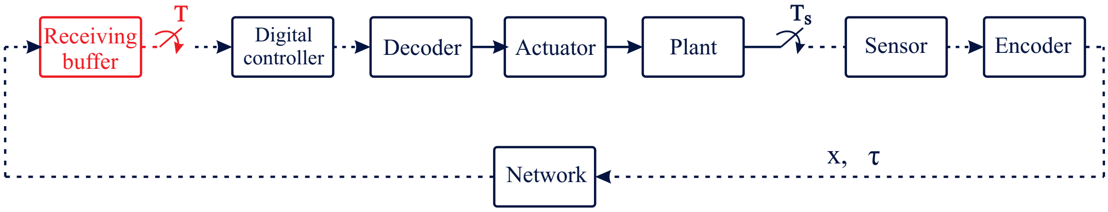


FIGURE 5.1: Single-channel networked control system

is labeled as outdated, the controller computes a control action based on prediction. However, if the plant is affected by disturbance, the control signal obtained in this way may introduce a tracking error which grows with every new delay variation. For this reason, we designed a model-based controller with a compensation block which is capable to correct the induced tracking error. A sufficient condition that guarantees the existence of the compensation signal is also provided. Simulations and numerical examples follow the theoretical presentation throughout the chapter. Brief concluding remarks are outlined in the last section.

5.1 Time-delay variation as a fault

Networked control system model that is considered in this section has controller collocated with the actuator (see Fig. 5.1). Such configuration allows us to neglect controller-to-actuator delays, which on the other side, can be tackled according to theoretical results outlined in Chapter 2 and the references therein.

Let us evoke briefly the general functioning of the NCS scheme depicted on Fig. 5.1. Ideally, at each sampling period control action should be updated based on the latest measurements acquired by the sensor. However, because it is shared by several (possibly many) nodes, the communication network may not be idle at the moment when it is required by the sensor. When that happens, data packets are hold until the network protocol grants permission for their transmission. For the most network protocols such induced delays are random and time-varying. Transmitted data are stored in the receiving buffer which is read by the controller at a higher frequency than the sampling frequency of the plant.

Control design becomes more involved with the presence of time-delays at the input. For example, a stabilizing controller for delay-free dynamics may not stabilize the same system affected by delay. The same holds for constant versus time-varying delay. For instance, let $\bar{\tau}[t_k] = (d[t_k]-1)T_s + \bar{\tau}_{is}[t_k]T$ and $\bar{\tau}[t_{k+1}] = \bar{\tau}_{is}[t_{k+1}]T$ (see Chapter 2 for the delay characterization) denote sensor-to-controller delays for two consecutive sampling periods for the NCS scheme which is depicted on Fig. 5.1. Rewriting discrete-time models with respect to $\bar{\tau}[t_k]$ and $\bar{\tau}[t_{k+1}]$ in the augmented state-space (see Chapter 2), we obtain the following dynamics:

$$\begin{bmatrix} x[t_{k+2}] \\ x[t_{k+1}] \end{bmatrix} = \Lambda(\bar{\tau}_{is}[t_{k+1}])\Lambda(\bar{\tau}_{is}[t_k]) \begin{bmatrix} x[t_k] \\ x[t_{k-d}] \end{bmatrix}.$$

It was shown in Chapter 2 that such dynamics can be unstable even for the stable matrices $\Lambda(\bar{\tau}_{is}[t_{k+1}])$ and $\Lambda(\bar{\tau}_{is}[t_k])$. As an illustration how time-delay can affect a networked control system we propose the following example.

Example 5.1. Mechanical system of a rolling ball and a tilting beam is shown on Fig. 5.2. Mathematical model of the mechanism is given by the following linearized differential equation (see Aström and Wittenmark [1997]):

$$\frac{d^2y(t)}{dt^2} = \frac{mgr^2}{J}\sin(\varphi(t)) \approx \frac{mgr^2}{J}\varphi(t),$$

where m, r and J stand for the mass, the radius and the moment of inertia of the ball respectively, while the tilting angle of the beam φ is regarded as the input.

Dynamics of the system can be fully described by a relative position and velocity of the ball, brought together in the form of a state vector $x(t) = [x_1(t)\ x_2(t)]^T = [y(t)\ \dot{y}(t)]^T$. We assume that control of the tilting angle is implemented via network that induces time-varying delays. Continuous-time state-space representation of this systems is given by:

$$\frac{dx(t)}{dt} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{mgr^2}{J} \end{bmatrix} \varphi(t - \tau(t)).$$

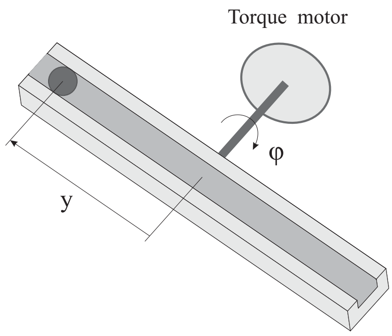


FIGURE 5.2: Mechanical system of a rolling ball and a beam

T_s	1 s
m	0.4 kg
r	0.03 m
K	[0.0271 0.0543]

TABLE 5.1: Numerical values

In order to simplify the example, we assume that all states are measurable and that control action is provided by a digital linear quadratic regulator $\varphi[t_k] = -Kx[t_k]$, which is designed to stabilize the plant when $\tau[t_k] = 0$, $\forall k \in \mathbb{Z}_+$.

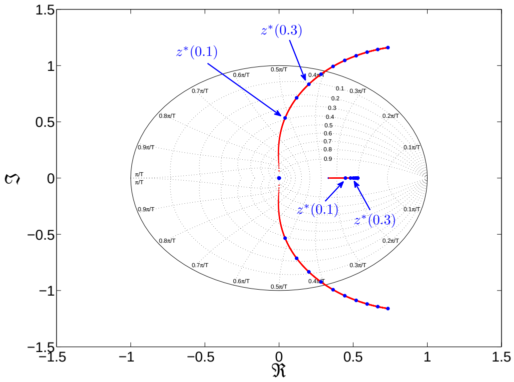
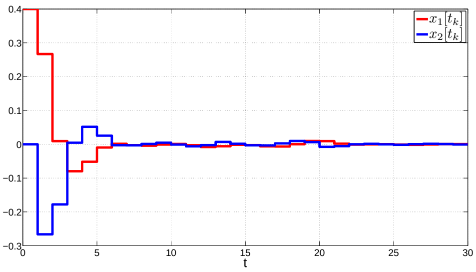
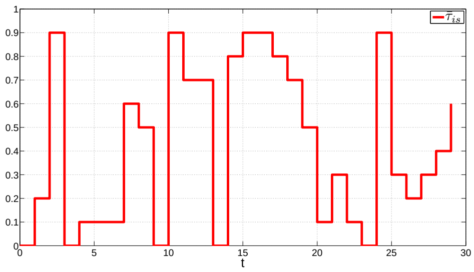


FIGURE 5.3: Root locus for $\tau \in \mathbb{R}_{[0,T_S]}$



(A) Closed-loop response

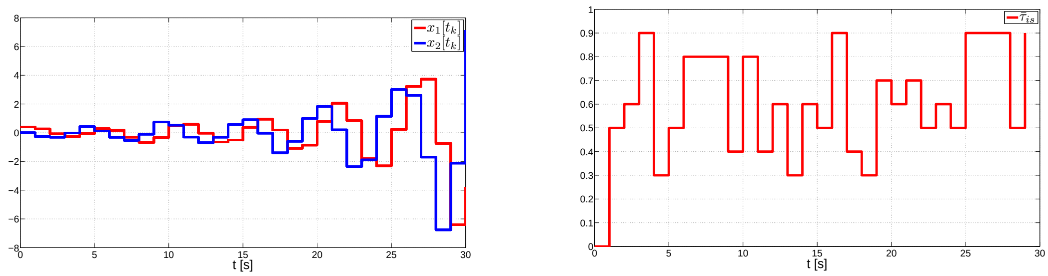


(B) Delay variation

FIGURE 5.4: Stable closed-loop response

For varying delays, the simulation results are shown on Fig. 5.4 and Fig. 5.5. It is clear that the closed-loop system may have significantly different behavior with respect to delay variation.

A *fault* is formally defined as a deviation of the system structure or the system parameters from the nominal specification (see e.g. Blanke [2003]). Following the same idea, one can consider a network-induced delay as a deviation in the communication channels with respect to the *nominal*, delay-free data transmission. The last statement represents the main approach that we examine in this chapter.



(A) Closed-loop response

(B) Delay variation

FIGURE 5.5: Unstable closed-loop response

5.2 Nominal NCS description

Let us consider the following LTI plant:

$$\dot{x}(t) = A_c x(t) + B_c u(t) + E_c \omega(t), \tag{5.1}$$

controlled over a shared communication network. It is assumed that the control action is provided by a controller which is collocated with the actuator. We denote by $x \in \mathbb{R}^n$ the state vector, by $u \in \mathbb{R}^m$ the control signal and by $\omega \in \mathbb{R}^p$ the bounded process noise. It is assumed that $\omega \in \mathbb{W}$, where $\mathbb{W} \subset \mathbb{R}^p$ is a symmetric C-set and that the matrices $A_c \in \mathbb{R}^{n\times n}$, $B_c \in \mathbb{R}^{n\times m}$ and $E_c \in \mathbb{R}^{n\times p}$ are constant.

A NCS is said to be nominal if the data transmission between the sensor and the controller is carried out without delays, i.e. if $\tau = 0$.

The plant (5.1) is controlled by a digital controller which is represented by the following piecewise-constant function:

$$u(t) = u[t_k], \quad \forall t \in [t_k, t_{k+1}), \quad k \in \mathbb{Z}_+. \tag{5.2}$$

Using zero-order hold sampling with the constant period $T_s = t_{k+1} - t_k$, the nominal discrete-time representation of (5.1) is given as:

$$x[t_{k+1}] = Ax[t_k] + Bu[t_k] + E\omega[t_k], \tag{5.3}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $E \in \mathbb{R}^{n\times p}$ are constant discrete-time equivalents of A_c, B_c and E_c (see Chapter 2 and (2.4)).

The control objective is for the state of the plant (5.3) to track a reference signal x_{ref} that satisfies the reference dynamics

$$x_{ref}[t_{k+1}] = Ax_{ref}[t_k] + Bu_{ref}[t_k]. \tag{5.4}$$

The nominal control signal is obtained according to the following equation:

$$u[t_k] = u_{ref}[t_k] + v[t_k], \tag{5.5}$$

where the reference control input u_{ref} can represent for instance a stabilizing control input when the reference dynamics (5.4) is not stable, while v represents a tracking control law which ensures convergence of the tracking error $z = x - x_{ref}$.

In order to compute the control signal (5.5), we consider either *state feedback* or *estimation state feedback*. Both cases will be often simultaneously considered by using a common notation. However, some notions are not completely compatible and for those cases the state feedback and the estimation feedback need to be examined separately.

5.2.1 State feedback and nominal control design

The state feedback architecture is depicted on Fig. 5.1.

The state vector is measured every T_s by a sensor which is assumed to be static (or with very fast dynamics relatively to the plant dynamics) and which satisfies the observation equation:

$$y[t_k] = Cx[t_k], \tag{5.6}$$

where $y \in \mathbb{R}^n$ and $C \in \mathbb{R}^{n\times n}$ such that $rank(C) = n$. Without loss of generality, we assume that $C = I_n$ whenever we consider the state feedback.

Remark 5.1. In the observation equation (5.6) we may also consider an additive measurement disturbance. Naturally, this disturbance would have to be bounded. Such an extension would not require significant changes in the subsequent theoretical consideration. Therefore, in order to simplify the presentation, we decided to consider only this simplified case. However, since it requires a particular theoretical development, measurement noise will be taken into account in the following chapter.

In order to design a controller which provides attenuation of negative effects caused by sensor-to-controller delays, in this work we often rely on the set-theoretic methods. Therefore, beside satisfying performance of the closed-loop system, it is also preferable for control action to rely on reduced computational effort (for instance with respect to invariant sets construction). Method that is presented here provides an active strategy[2]

[2]By active we assume the capability for taking an appropriate correcting action whenever a delay is detected.

to tackle random time-varying delays and it is independent on a chosen tracking control law. Therefore, in order to simplify the presentation, we employ the LQR as a generic LTI system stabilization method.

The following hypothesis holds throughout this chapter.

Hypothesis 5.1. The pair (A, B) is controllable.

Let us consider (5.5) and the following tracking control law:

$$v[t_k] = -K(x[t_k] - x_{ref}[t_k]). \tag{5.7}$$

The gain $K \in \mathbb{R}^{m\times n}$ is obtained by minimizing the following cost function :

$$J(z, v) = z^T Qz + v^T Rv + (Az + Bv)^T P (Az + Bv), \tag{5.8}$$

where $z = x - x_{ref}$ is the tracking error while Q and R are positive definite cost matrices. For a given Q and R, matrices P and K are computed from the following algebraic Riccati equation:

$$\begin{aligned} K &= (R + B^T PB)^{-1} B^T PA \\ P &= A^T PA + Q - K^T (R + B^T PB)K. \end{aligned} \tag{5.9}$$

Using (5.9) the cost function (5.8) can be rewritten as:

$$J(z, v) = z^T Pz. \tag{5.10}$$

From (5.7), (5.8) and (5.10), we obtain Lyapunov's matrix equation:

$$(A - BK)^T P(A - BK) + P = -(Q + K^T RK).$$

It is well known that if and only if Hypothesis 5.1 holds, for any positive definite Q and R there exists a unique positive definite matrix P and a control gain K such that all poles of $A - BK$ are inside the unit circle (see Kalman [1960]).

For (5.7), (5.3) and (5.4), the closed-loop performance of the nominal NCS is determined by the following tracking error dynamics:

$$z[t_{k+1}] = (A - BK)\, z[t_k] + E\omega[t_k]. \tag{5.11}$$

5.2.2 Estimated state feedback

The estimated state feedback architecture is depicted on Fig. 5.6.

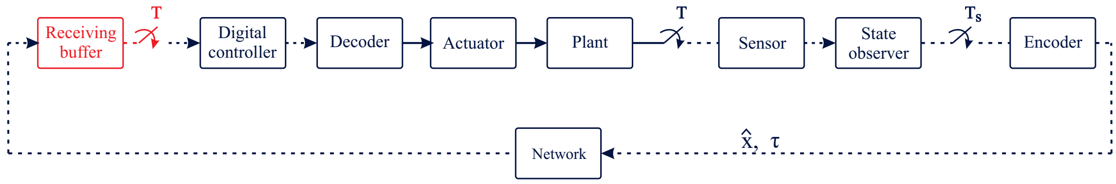


FIGURE 5.6: Single-channel networked control system

If the state vector is not completely measurable, we employ a state observer in order to obtain its estimation. It is assumed that the state observer is collocated with the sensor. Such a choice can be justified by an increased number of *smart sensors* with embedded processing unit, capable to locally process the raw measurements and to transmit the local estimates (see Xu and Hespanha [2005]).

It was already stated (see Chapter 2) that, in the presence of delay, control input is subject to a variation within the sampling period T_s. In order to obtain proper estimations of the state vector and to avoid using a remote estimator, we need to provide the state observer with the same control input as the plant. Therefore, the state observer is designed with respect to the inter-sampling dynamics of the process.
Let us assume that the output of the process is measured at a faster rate, say with the period T, where $T_s = NT$ for sufficiently large $N \in \mathbb{Z}^+$, while the state estimate is still transmitted to the controller each T_s. The sensor is assumed to be static (or a sensor with very fast dynamics relatively to the plant dynamics) which satisfies the observation equation:

$$y[t_k + iT] = Cx[t_k + iT], \quad i \in \mathbb{Z}_{[0,(N-1)]}, \tag{5.12}$$

where $y \in \mathbb{R}^r$ and $C \in \mathbb{R}^{r\times n}$ such that $rank(C) < n$.

Remark 5.2. Without loss of any generality, we decide to consider the simplified model of the output (5.12), i.e., a noise-free sensor (see Remark 5.1). In the single feedback case such an assumption does not pose any particular problem. However, this hypothesis needs to be carefully addressed in the case of redundant sensor architecture that will be consider in the subsequent chapter.

Using the constant inter-sampling period T, the corresponding nominal plant dynamics (5.3) is given by:

$$x[t_k + (i+1)T] = \tilde{A}x[t_k + iT] + \tilde{B}u[t_k] + \tilde{E}\omega[t_k], \quad i \in \mathbb{Z}_{[0,(N-1)]}, \tag{5.13}$$

where $\tilde{A} \in \mathbb{R}^{n\times n}$, $\tilde{B} \in \mathbb{R}^{n\times m}$ and $\tilde{E} \in \mathbb{R}^{n\times p}$ are constant matrices (see Chapter 2). Notice that it has been assumed that ω is constant between two consecutive samplings. Such an assumption can be easily relaxed according to the discussion outlined in Section 3.2.

The following assumption holds.

Hypothesis 5.2. The pair $(\tilde{A}, C)$ is observable.

The observed state vector is represented by the following discrete-time dynamics:

$$\hat{x}[t_k + (i+1)T] = \tilde{A}\hat{x}[t_k + iT] + \tilde{B}u[t_k] + L\left(y[t_k + iT] - C\hat{x}[t_k + iT]\right), \quad i \in \mathbb{Z}_{[0,N-1]}, \tag{5.14}$$

where $L \in \mathbb{R}^{n\times r}$ is the observer gain designed to guarantee the placement of the poles of $\tilde{A} - LC$ inside the unit disc (always possible by Hypothesis 5.2).

The relevance of the estimated state vector is evaluated by the estimation error ($\tilde{x} = x - \hat{x}$) which is obtained from (5.13) and (5.14) as:

$$\tilde{x}[t_k + (i+1)T] = \left(\tilde{A} - LC\right)\tilde{x}[t_k + iT] + \tilde{E}\omega[t_k], \quad i \in \mathbb{Z}_{[0,N-1]}. \tag{5.15}$$

It is worth mentioning that (5.15) does not depend on delay value since (5.14) is always provided by the same input as (5.13). Therefore, for a stable $\tilde{A} - LC$, the estimation error (5.15) is bounded. Moreover, if $\omega \in \mathcal{W}$, (5.15) admits a robust positively invariant set $\tilde{\mathcal{X}}$ (see Chapter 3) such that:

$$\tilde{x}[t_k + (i+1)T] \in \tilde{\mathcal{X}}, \quad \forall \tilde{x}[t_k + iT] \in \tilde{\mathcal{X}}, \quad \forall \omega[t_k] \in \mathcal{W}. \tag{5.16}$$

Taking into account the separation principle (for the state and the estimation feedback), the tracking control law from (5.5) can be rewritten for the estimated state feedback case as:

$$v[t_k] = -K\left(\hat{x}[t_k] - x_{ref}[t_k]\right). \tag{5.17}$$

Nominal dynamics of the state observer with respect to T_s is given by the following discrete-time equation:

$$\hat{x}[t_{k+1}] = A\hat{x}[t_k] + Bu[t_k] + \sum_{i=0}^{N-1} \tilde{A}^i LC\tilde{x}[t_k + (N-1-i)T]. \tag{5.18}$$

Replacing (5.15) in (5.18), one has:

$$\hat{x}[t_{k+1}] = A\hat{x}[t_k] + Bu[t_k] + F\tilde{x}[t_k] + G\omega[t_k], \tag{5.19}$$

where

$$
\begin{aligned}
F &= \begin{bmatrix} \tilde{A}^{N-1}LC & \tilde{A}^{N-1}LC & \dots & LC \end{bmatrix} \begin{bmatrix} I_{n\times n} \\ \tilde{A}-LC \\ \vdots \\ (\tilde{A}-LC)^{N-1} \end{bmatrix}, \\
G &= \begin{bmatrix} \tilde{A}^{N-1}LC & \tilde{A}^{N-1}LC & \dots & LC \end{bmatrix} \begin{bmatrix} 0_{n\times p} \\ \tilde{E} \\ \vdots \\ \sum_{i=0}^{N-2}(\tilde{A}-LC)^{i}\tilde{E} \end{bmatrix}.
\end{aligned} \tag{5.20}
$$

Since the tracking error z is not measurable, one needs to define the estimation tracking error $\hat{z} = \hat{x} - x_{ref}$. By using (5.4), (5.5), (5.17) and (5.19), the estimation tracking error is determined according to the following difference equation:

$$
\hat{z}[t_{k+1}] = (A - BK)\,\hat{z}[t_k] + F\tilde{x}[t_k] + G\omega[t_k], \tag{5.21}
$$

If $A - BK$ is Schur and $\tilde{x} \in \tilde{\mathcal{X}}$, then (5.21) admits a robust positively invariant set $\hat{\mathcal{Z}} \subset \mathbb{R}^n$ such that:

$$
\hat{z}[t_{k+1}] \in \hat{\mathcal{Z}}, \quad \forall \hat{z}[t_k] \in \hat{\mathcal{Z}}, \quad \forall \tilde{x}[t_k] \in \tilde{\mathcal{X}}, \quad \forall \omega[t_k] \in \mathcal{W}. \tag{5.22}
$$

Whenever the results are independent on explicit feedback information, we will use θ as the common notation in order to denote the controller's input, i.e., $\theta = x$ or $\theta = \hat{x}$. For the nominal NCS, θ is a piecewise-constant function such that

$$
\theta(t) = \theta[t_k], \quad \forall t \in [t_k, t_{k+1}), \quad k \in \mathbb{Z}_+. \tag{5.23}
$$

5.3 Model-based controller

The following assumption holds throughout this chapter.

Hypothesis 5.3. The pair $(\tilde{A}, \tilde{B})$ is controllable.

In order to tackle network-induced sensor-to-controller delays, a classical approach employs a *model-based controller* (see Montestruque and Antsaklis [2003], Witrant et al. [2007] and Zhang et al. [2001]). Such a controller utilizes mathematical model of the process in order to estimate current state vector from available outdated measurements.

In this section, we consider a switching digital controller with two control loops (see Fig. 5.7). If the controller is provided with the outdated measurements, the control

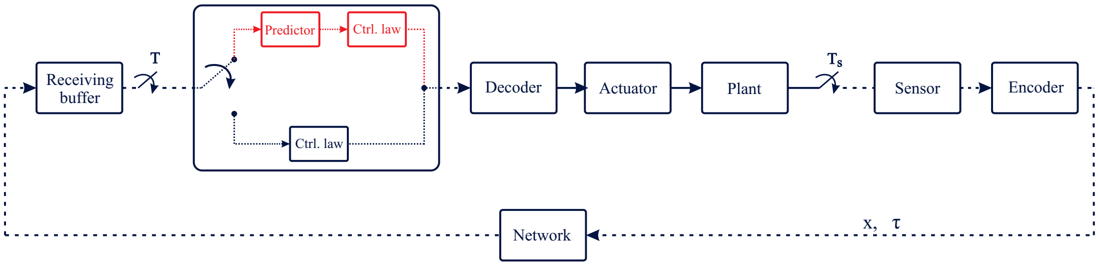


FIGURE 5.7: Model-based controller

action is generated based on information provided by the predictor (upper control loop on Fig. 5.7). On the other side, when up-to-date measurements are available, the controller is switched to the second control loop.

Let us regard the time interval $t \in [t_k, t_{k+1})$ and denote by $\theta[t_{k-d[t_k]}]$ the most recently delivered feedback information to the controller. Using $\theta[t_{k-d[t_k]}]$, along with the mathematical model of the plant, model-based controller calculates the following prediction:

$$\theta_{t_k|t_{k-d[t_k]}} = A^{d[t_k]}\theta[t_{k-d[t_k]}] + \sum_{q=0}^{d[t_k]-1} A^q B u_{t_{k-q-1}|t_{k-d[t_k]}}, \tag{5.24}$$

where

$$u_{t_{k-q-1}|t_{k-d[t_k]}} = u_{ref}[t_{k-q-1}] - K\left(\theta_{t_{k-q-1}|t_{k-d[t_k]}} - x_{ref}[t_{k-i-1}]\right).$$

We use the notation $\theta_{t_k|t_{k-d[t_k]}}$ in order to point out that this is a prediction of $\theta[t_k]$ which is obtained based on $\theta[t_{k-d[t_k]}]$. Obviously, the one will have $\theta_{t_{k-q}|t_{k-q}} = \theta[t_{k-q}]$. Notice that the reference input u_{ref} and the reference state vector x_{ref} are always available for the control signal computation.

In the presence of disturbance, it is clear that (5.24) may not provide sufficiently accurate information on the real plant's behavior as that would be the case with the sensor.

Let $\bar{\tau}[t_k] = (d[t_k] - 1)\, T_s + \bar{\tau}_{is}[t_k]T$, where $\bar{\tau}_{is}[t_k] \in \mathbb{Z}_{[0,N-1]}$. Thus,

$$u_{t_k|t_{k-d[t_k]}} = u_{ref}[t_k] - K\left(\theta_{t_k|t_{k-d[t_k]}} - x_{ref}[t_k]\right), \tag{5.25}$$

determines the control signal on the interval $t \in \mathbb{R}_{[t_k, t_k+\bar{\tau}_{is}[t_k]T)}$.

For the control signal (5.25), the state vector is obtained as:

$$x[t_k + \bar{\tau}_{is}[t_k]T] = \tilde{A}^{\bar{\tau}_{is}[t_k]}x[t_k] + \sum_{i=0}^{\bar{\tau}_{is}[t_k]-1} \tilde{A}^i \tilde{B} u_{t_k|t_{k-d[t_k]}} + \sum_{i=0}^{\bar{\tau}_{is}[t_k]-1} \tilde{A}^i \tilde{E}\omega[t_k]. \tag{5.26}$$

When $\theta[t_k]$ is delivered to the buffer (at $t \in \mathbb{R}_{(t_k+(\bar{\tau}_{is}[t_k]-1)T, t_k+\bar{\tau}_{is}[t_k]T]}$), the control action is updated at $t = t_k + \bar{\tau}_{is}[t_k]T$. The control signal becomes $u[t_k]$, and it is constant on the interval $t \in \mathbb{R}_{[t_k+\bar{\tau}_{is}[t_k]T, t_{k+1})}$. The corresponding state vector at t_{k+1} is given as:

$$x[t_{k+1}] = Ax[t_k] + \sum_{i=N-\bar{\tau}_{is}[t_k]}^{N-1} \tilde{A}^i \tilde{B} u_{t_k|t_{k-d[t_k]}} + \sum_{i=0}^{N-\bar{\tau}_{is}[t_k]-1} \tilde{A}^i \tilde{B} u[t_k] + E\omega[t_k], \tag{5.27}$$

where $A = \tilde{A}^N$, $E = \sum_{i=0}^{N-1} \tilde{A}^i \tilde{E}$.

A possible limitation of this approach could be the fact that two assumptions need to be imposed: sufficiently accurate model of the plant is known and for each sensor-to-controller delay, the corresponding interval where such a delay resides is identified. If one of these assumptions does not hold, the overall functioning of the control strategy may be compromised. While the problem of delay detection is considered in Section 5.5, here we mainly examine issues with respect to process uncertainties. Therefore, in this section we assume that all sensor-to-controller delays are known to the controller.

Since (5.24) does not provide information on the process noise, using this signal instead of the corresponding measurements, may induce a tracking error that grows with every subsequent control based on prediction. In order to define upper bounds on the tracking error with respect to delay, let us define the following prediction error:

$$\varepsilon_{t_k|t_{k-d[t_k]}} = \theta[t_k] - \theta_{t_k|t_{k-d[t_k]}}. \tag{5.28}$$

Using (5.28) in (5.25) the control signal becomes:

$$u_{t_k|t_{k-d[t_k]}} = u[t_k] + K\varepsilon_{t_k|t_{k-d[t_k]}}. \tag{5.29}$$

For (5.29), the closed loop system (5.27) is written as:

$$x[t_{k+1}] = Ax[t_k] + Bu[t_k] + E\omega[t_k] + \sum_{i=N-\bar{\tau}_{is}[t_k]}^{N-1} \tilde{A}^i \tilde{B} K \varepsilon_{t_k|t_{k-d[t_k]}}, \tag{5.30}$$

where $B = \sum_{i=0}^{N-1} \tilde{A}^i \tilde{B}$.

5.3.1 Bounds on tracking error for the state feedback case

Let us first consider the state feedback case, i.e., $\theta = x$. Using (5.24) and (5.28), we define the one-step prediction error as:

$$\varepsilon_{t_k|t_{k-1}} = x[t_k] - Ax[t_{k-1}] - Bu[t_{k-1}] = E\omega[t_{k-1}] \in \mathcal{E}_1, \tag{5.31}$$

where $\mathcal{E}_1 = E\mathcal{W}$. It is worth mentioning that $\varepsilon_{t_{k-i}|t_{k-j}} \in \mathcal{E}_1$ holds $\forall i, j \in \mathbb{Z}_+$ such that $i < j$ and $j - i = 1$. Since $\mathcal{W}$ is a symmetric C-set, $\mathcal{E}_1$ is a symmetric C-set as well (see e.g. Ziegler [1995]). In the general case, prediction error is confined in a set which is constructed according to the following recursive formula:

$$\varepsilon_{t_{k-i}|t_{k-j}} \in \mathcal{E}_{j-i} = \left\{ \bigoplus_{q=0}^{j-i-1} A^q E\mathcal{W} \right\} \oplus \left\{ \bigoplus_{q=1}^{j-i-1} A^{q-1} BK\mathcal{E}_{j-i-q} \right\}, \tag{5.32}$$

with $i, j \in \mathbb{Z}_+$ such that $i < j$ and $j - i \geq 2$. The expression (5.32) is derived from (5.24) and (5.28). Notice that the prediction error is bounded for $\tau \in \mathbb{R}_{[0,\tau_{max}]}$, $\tau_{max} < \infty$.

For $\theta = x$ the nominal tracking error dynamics is obtained as in (5.11). If the matrix $A - BK$ is Schur, then there exists a robust positively invariant set $\mathcal{Z}$ (see Chapter 4) such that

$$z[t_{k+1}] \in \mathcal{Z}, \quad \forall z[t_k] \in \mathcal{Z}, \quad \forall \omega[t_k] \in \mathcal{W}. \tag{5.33}$$

Such invariance property of the set $\mathcal{Z}$ does not hold anymore if the system in the closed-loop is affected by delays (see (5.30)). However, under certain conditions, it is possible to determine upper bounds for the tracking error dynamics with respect to delay parameter.

Proposition 5.1. *Let* $\tau \in \mathbb{R}_{((d[t_k]-1)T_s, d[t_k]T_s]}$. *If* $x[t_{k-d[t_k]+1}] \in \left\{ x_{ref}[t_{k-d[t_k]+1}] \right\} \oplus \mathcal{Z}$, *then*

$$x[t_{k+1}] \in \{x_{ref}[t_{k+1}]\} \oplus \mathcal{Z} \oplus A \left\{ \bigoplus_{q=0}^{d[t_k]-2} A^q BK\mathcal{E}_{d[t_k]-1-q} \right\} \oplus \left\{ \bigoplus_{i=N-\bar{\tau}_{is}[t_k]}^{N-1} \tilde{A}^i \tilde{B} K\mathcal{E}_{d[t_k]} \right\}. \tag{5.34}$$

Proof. If $\tau \in \mathbb{R}_{((d[t_k]-1)T_s, d[t_k]T_s]}$, then $\bar{\tau}[t_k] = ((d[t_k]-1)T_s + \bar{\tau}_{is}[t_k]T)$, $\bar{\tau}_{is}[t_k] \in \mathbb{Z}_{[0,N-1]}$.

Using (5.25) and considering the state vector evolution backward in time, i.e., from $t = t_{k+1}$ back to $t = t_{k-d[t_k]+2}$, we have:

$$\begin{aligned} x[t_{k+1}] &= Ax[t_k] + Bu[t_k] + E\omega[t_k] + \sum_{i=N-\bar{\tau}_{is}[t_k]}^{N-1} \tilde{A}^i \tilde{B} K\varepsilon_{t_k|t_{k-d[t_k]}}, \\ x[t_k] &= Ax[t_{k-1}] + Bu[t_{k-1}] + E\omega[t_{k-1}] + BK\varepsilon_{t_{k-1}|t_{k-d[t_k]}}, \\ &\vdots \\ x[t_{k-d[t_k]+2}] &= Ax[t_{k-d[t_k]+1}] + Bu[t_{k-d[t_k]+1}] + E\omega[t_{k-d[t_k]+1}] + BK\varepsilon_{t_{k-d[t_k]+1}|t_{k-d[t_k]}}. \end{aligned} \tag{5.35}$$

Substituting the previous series of equations backward, until we get to (5.35), the state vector at $t = t_{k+1}$ is obtained as:

$$\begin{aligned} x[t_{k+1}] = A^{d[t_k]}x[t_{k-d[t_k]+1}] &+ \sum_{q=0}^{d[t_k]-1} A^q Bu[t_{k-q}] + \sum_{q=0}^{d[t_k]-1} A^q E\omega[t_{k-q}] \\ &+ A\sum_{q=0}^{d[t_k]-2} A^q BK\varepsilon_{t_{k-q-1}|t_{k-d[t_k]}} + \sum_{i=N-\bar{\tau}_{is}[t_k]}^{N-1} \tilde{A}^i\tilde{B}K\varepsilon_{t_k|t_{k-d[t_k]}}; \end{aligned} \tag{5.36}$$

If $x[t_{k-d[t_k]+1}] \in \left\{x_{ref}[t_{k-d[t_k]+1}]\right\} \oplus \mathcal{Z}$, where $\mathcal{Z}$ is an invariant set with respect to the nominal dynamics, then, by using (5.32), one obtains (5.34). $\square$

5.3.2 Bounds on tracking error for the estimated state feedback case

In this subsection we determine bounds on the tracking error when $u = u_{ref} - K\left(\hat{x} - x_{ref}\right)$ and $\tau \neq 0$.

Let us recall the nominal ($\tau = 0$) closed-loop tracking error system for the estimated state feedback case, which is obtained from (5.3), (5.4) and (5.17):

$$z[t_{k+1}] = (A - BK)\, z[t_k] + E\omega[t_k] + BK\tilde{x}[t_k], \tag{5.37}$$

where we used the fact that $\hat{z} = z - \tilde{x}$.

When $\tau \neq 0$ the process is controlled based on prediction. Therefore, prediction error arises. By using (5.24), (5.28) and (5.14), one can define one-step prediction error as:

$$\begin{aligned} \varepsilon_{t_k|t_{k-1}} = \hat{x}[t_k] - \hat{x}_{t_k|t_{k-1}} &= \sum_{i=0}^{N-1} \tilde{A}^i LC\tilde{x}[t_k + (N-1-i)T] \\ &= F\tilde{x}[t_{k-1}] + G\omega[t_{k-1}], \end{aligned} \tag{5.38}$$

where F and G are obtained according to (5.20). Moreover, if $\tilde{x}[t_{k-1}] \in \tilde{\mathcal{X}}$ and $\omega[t_{k-1}] \in \mathcal{W}$, then $\varepsilon_{t_k|t_{k-1}} \in \mathcal{E}_1$, where

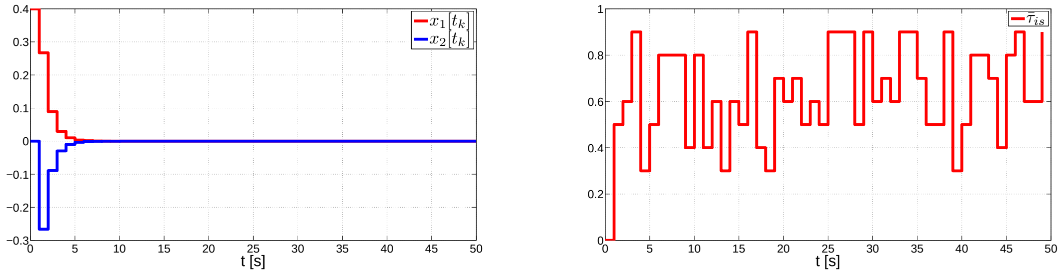
$$\mathcal{E}_1 = F\tilde{\mathcal{X}} \oplus G\mathcal{W}. \tag{5.39}$$

The same holds for each $\varepsilon_{t_{k-i}|t_{k-j}}$ with $i < j$ and $j - i = 1$.

If $\tilde{x}[t_{k-q}] \in \tilde{\mathcal{X}}$, then

$$\varepsilon_{t_{k-i}|t_{k-j}} \in \mathcal{E}_{j-i} = \left\{\bigoplus_{q=0}^{j-i-1} A^q\tilde{\mathcal{X}}\right\} \oplus \left\{\bigoplus_{q=1}^{j-i-1} A^{q-1}BK\mathcal{E}_{j-i-q}\right\}, \tag{5.40}$$

with $i, j \in \mathbb{Z}_+$ such that $i < j$ and $j - i \geq 2$.



(A) Closed-loop response (B) Delay variation

FIGURE 5.8: Closed-loop response when $\omega = 0$

The nominal tracking error system (5.37) admits a robust positively invariant set, say $\mathcal{Z}$, if $A - BK$ is Schur, $\forall \omega \in \mathcal{W}$ and $\forall \tilde{x} \in \tilde{\mathcal{X}}$, where $\mathcal{W}$ and $\tilde{\mathcal{X}}$ are $C-$sets. However, one can notice that, due to prediction error, positive invariance of $\mathcal{Z}$ may not hold. However, as it was done in the state feedback case, it is possible to determine upper bounds for the tracking error dynamics when the system is affected by delay.

Proposition 5.2. *Let $\tau \neq 0$ and $\tau \in \mathbb{R}_{((d[t_k]-1)T_s, d[t_k]T_s]}$, where $d = \left\lceil \frac{\tau}{T_s} \right\rceil$. If $x[t_{k-d[t_k]+1}] \in \left\{x_{ref}[t_{k-d[t_k]+1}]\right\} \oplus \mathcal{Z}$, then*

$$x[t_{k+1}] \in \{x_{ref}[t_{k+1}]\} \oplus \mathcal{Z} \oplus A \left\{ \bigoplus_{q=0}^{d[t_k]-2} A^q BK \mathcal{E}_{d[t_k]-1-q} \right\} \oplus \left\{ \bigoplus_{N-\bar{\tau}_{is}[t_k]}^{N-1} \tilde{A}^i \tilde{B} K \mathcal{E}_{d[t_k]} \right\}, \tag{5.41}$$

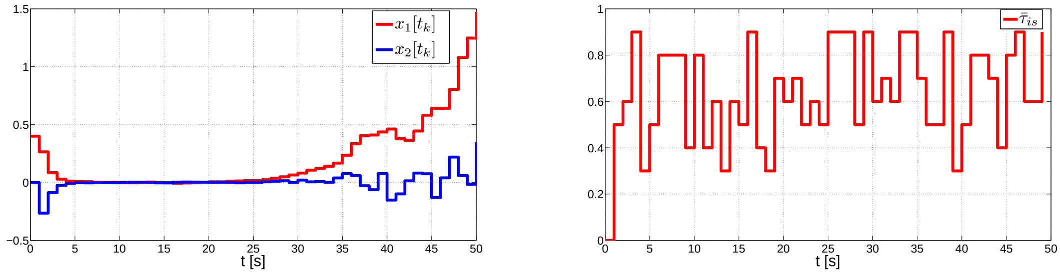
where the sets $\mathcal{E}$ are determined according to (5.38).

Proof. The proof is similar as the proof for Proposition 5.1, and therefore is omitted here. □

Through the following example we clarify how model-based control strategy performs in the presence of a random delay variation (delays are assumed to be smaller then the sampling period) and the additive disturbance.

Example 5.2. We get back to the system of a rolling ball and a beam that was considered before. Now it is assumed that state vector is affected by a bounded additive disturbance $\omega \in \mathbb{W} \subset \mathbb{R}^2$ such that $\|\omega\|_\infty \leq 0.004$.

$$\frac{dx(t)}{dt} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{mgr^2}{J} \end{bmatrix} \varphi(t - \tau(t)) + \omega(t). \tag{5.42}$$



(A) Closed-loop response

(B) Delay variation

FIGURE 5.9: Closed-loop response when $\| \omega \|_\infty \leq 0.004$

Let us first consider the case when $\omega(t) = 0$, $\forall t \in \mathbb{R}_+$. Using the previously discussed model-based controller, the closed-loop response of the system is shown on Fig. 5.8. Notice that this system was unstable for the same delay pattern and the LQR controller (see Fig. 5.5)

For $\|\omega\|_\infty \leq 0.004$, $\forall t \in \mathbb{R}_+$ response of the closed-loop dynamics controlled by the model-based controller is shown on Fig. 5.9. Such a response confirms that, in general, model-based controller cannot provide satisfying control performance in the presence of disturbance.

Based on the previous example, one can conclude that performance of a system controlled by a model-based controller can be significantly compromised in the presence of even relatively small perturbations. Therefore we propose a strategy which employs a model-based controller, accompanied by a delay compensator, which preserves tracking error within pre-defined bounds.

5.4 Model-based controller with active delay compensation

Design of the delay compensator exploits the idea that a delay occurrence can be considered as a fault with regard to the nominal dynamics. In order to reduce the tracking error in this case, we design a FDI mechanism (in Section 5.5) and the corresponding control reconfiguration. This subsection deals with the control reconfiguration.

Block diagram of the considered control strategy is presented on Fig. 5.10.

Let $\bar{\tau}[t_k] = (d[t_k] - 1)\, T_s + \bar{\tau}_{is}[t_k]T$. For $\forall t \in \mathbb{R}_{[t_k, t_k + \bar{\tau}_{is}[t_k]T)}$ the control signal is generated based on prediction (5.24). Hence, the state vector at $t = t_k + \bar{\tau}_{is}[t_k]T$ is the same as in

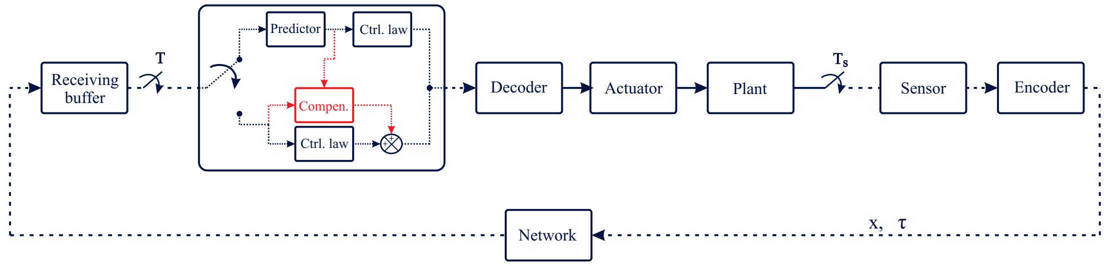


FIGURE 5.10: Model-based controller with active delay compensation

(5.26). On the other side, for available $\theta[t_k]$ the state vector at t_{k+1} is given as:

$$
\begin{aligned}
x[t_{k+1}] &= \tilde{A}^N x[t_k] + \sum_{i=N-\bar{\tau}_{is}[t_k]}^{N-1} \tilde{A}^i \tilde{B} u_{t_k|t_{k-d[t_k]}} + \sum_{i=0}^{N-1} \tilde{A}^i \tilde{E}\omega[t_k] \\
&+ \sum_{i=0}^{N-\bar{\tau}_{is}[t_k]-1} \tilde{A}^i \tilde{B}\left(u[t_k] + \sigma[t_k + (N-1-i)T]\right) \\
&= Ax[t_k] + \sum_{i=N-\bar{\tau}_{is}[t_k]}^{N-1} \tilde{A}^i \tilde{B}\left\{u_{ref}[t_k] - K\left(\theta_{t_k|t_{k-d[t_k]}} - x_{ref}[t_k]\right)\right\} \\
&+ E\omega[t_k] + \sum_{i=0}^{N-\bar{\tau}_{is}[t_k]-1} \tilde{A}^i \tilde{B}\left\{u_{ref}[t_k] - K\left(\theta[t_k] - x_{ref}[t_k]\right)\right\} \\
&+ \sum_{i=0}^{N-\bar{\tau}_{is}[t_k]-1} \tilde{A}^i \tilde{B}\sigma[t_k + (N-1-i)T],
\end{aligned} \tag{5.43}
$$

where

$$
\sigma_k = \left[\sigma[t_k + \bar{\tau}_{is}[t_k]T]^T \quad \dots \quad \sigma[t_k + (N-1)T]^T\right] \in \mathbb{R}^{(N-\bar{\tau}_{is}[t_k])m} \tag{5.44}
$$

is a compensation vector which is designed in order to compensate for the tracking error which is induced due to the prediction error (5.28).

Using (5.28) in (5.43) one can obtain

$$
\begin{aligned}
x[t_{k+1}] &= Ax[t_k] + Bu[t_k] + E\omega[t_k] + \sum_{i=N-\bar{\tau}_{is}[t_k]}^{N-1} \tilde{A}^i \tilde{B} K \varepsilon_{t_k|t_{k-d[t_k]}} \\
&+ \sum_{i=0}^{N-\bar{\tau}_{is}[t_k]-1} \tilde{A}^i \tilde{B}\sigma[t_k + (N-1-i)T].
\end{aligned} \tag{5.45}
$$

5.4.1 State feedback compensation

In order to determine the proper compensation vector (5.44), we need to identify which information is available to the controller at the moment when up-to-date data are delivered. Regarding the system (5.45), one can notice that for delays smaller than the sampling period, such a compensation vector can be computed from the following linear equation:

$$\sum_{i=0}^{N-\bar{\tau}_{is}[t_k]-1} \tilde{A}^i \tilde{B} \sigma[t_k + (N-1-i)T] = -\left(\sum_{i=N-\bar{\tau}_{is}[t_k]}^{N-1} \tilde{A}^i \tilde{B} K \varepsilon_{t_k|t_{k-d[t_k]}} \right). \tag{5.46}$$

For delays that are larger then the sampling period, we also need to take into account the tracking error which is caused by delay propagation during several sampling periods. This tracking error is incorporated in the current state vector. From the equation (5.45) we can notice that $\varepsilon_{t_k|t_{k-d[t_k]}}$ becomes available with the receiving of the most recent information from the plant. The same holds also for $x[t_k]$ and $u[t_k]$. Hence, we can determine the compensation vector (5.44) as:

$$\sum_{i=0}^{N-\bar{\tau}_{is}[t_k]-1} \tilde{A}^i \tilde{B} \sigma[t_k+(N-1-i)T] = x_{ref}[t_{k+1}] - Ax[t_k] - Bu[t_k] - \sum_{i=N-\bar{\tau}_{is}[t_k]}^{N-1} \tilde{A}^i \tilde{B} K \varepsilon_{t_k|t_{k-d[t_k]}}, \tag{5.47}$$

where we did not take into account additive disturbance since this information is not available at the moment. Moreover, we set $\omega = 0$ due to the assumption that the process noise is zero mean. If this assumption does not hold, then one should take into consideration the mean value of ω in (5.47). If there exists a solution to (5.47), then, by using (5.47) in (5.45), one obtains:

$$x[t_{k+1}] \in \{x_{ref}[t_{k+1}]\} \oplus E\mathcal{W} \subset \{x_{ref}[t_{k+1}]\} \oplus \mathcal{Z}. \tag{5.48}$$

Sufficient condition for the existence of $\bar{\sigma}$ are discussed in Subsection 5.4.3.

5.4.2 Estimated state feedback compensation

Let $\theta = \hat{x}$. Differently from the state feedback case, here the controller does not have information on the state vector, but on its estimation. From the assumption that the process and the observer are provided by the same input signal, we have that the estimation error (5.15) is confined in the robust positively invariant set $\tilde{\mathcal{X}}$ after some transition period.

For $\bar{\tau}[t_k] = (d[t_k] - 1)\, T_s + \bar{\tau}_{is}[t_k]T$, we have:

$$\begin{aligned}\hat{x}[t_{k+1}] = \underbrace{A\hat{x}[t_k] + Bu[t_k] + \sum_{i=0}^{N-1} \tilde{A}^i LC\tilde{x}[t_k + (N-1-i)T]}_{\text{Nominal dynamics}} \\ + \sum_{i=N-\bar{\tau}_{is}[t_k]}^{N-1} \tilde{A}^i \tilde{B} K \varepsilon_{t_k|t_{k-d[t_k]}} + \sum_{i=0}^{N-\bar{\tau}_{is}[t_k]-1} \tilde{A}^i \tilde{B} \sigma[t_k + (N-1-i)T].\end{aligned} \tag{5.49}$$

At $t = t_k + \bar{\tau}_{is}[t_k]T$ controller receives information on $\hat{x}[t_k]$ and consequently on $\varepsilon_{t_k|t_{k-d[t_k]}}$. Using available information, one can compute the following compensation vector:

$$\sum_{i=0}^{N-\bar{\tau}_{is}[t_k]-1} \tilde{A}^i \tilde{B} \sigma[t_k + (N-1-i)T] = x_{ref}[t_{k+1}] - A\hat{x}[t_k] - Bu[t_k] - \left(\sum_{i=N-\bar{\tau}_{is}[t_k]}^{N-1} \tilde{A}^i \tilde{B} K \varepsilon_{t_k|t_{k-d[t_k]}} \right). \tag{5.50}$$

Combining (5.49) and (5.50) one obtains:

$$\hat{z}[t_{k+1}] = \sum_{i=0}^{N-1} \tilde{A}^i LC\tilde{x}[t_k + (N-1-i)T] = F\tilde{x}[t_k] + G\omega[t_k]. \tag{5.51}$$

Therefore, $\hat{z}[t_{k+1}] \in \hat{\mathcal{Z}}$, $\forall \tilde{x}[t_k] \in \tilde{\mathcal{X}}$ and $\forall \omega[t_k] \in \mathcal{W}$.

5.4.3 Stability analysis

It is clear that, in the general case, solutions of the linear equations (5.47) and (5.50) may not be feasible. Therefore, we provide the following sufficient condition that guarantees the existence of a compensation vector. This result holds for both closed-loop system configurations, i.e., for state and estimated state feedback.

Let us introduce the following definition (see Datta [2004]).

Definition 5.1. *Let $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ form a controllable pair (A, B). The controllability index μ is defined as the least integer such that*

$$rank(C) = rank\left(\begin{bmatrix} B & AB & \dots & A^{\mu-1}B \end{bmatrix}\right) = n$$

Theorem 5.1. *Let $\mu \in \mathbb{Z}^+$ denote the controllability index of the pair $(\tilde{A}, \tilde{B})$. Then, the linear equations (5.47) and (5.50) are consistent, i.e., they admit at least one solution for any vector on the right-hand side, if $\bar{\tau}_{is}[t_k] \leq N - \mu$.*

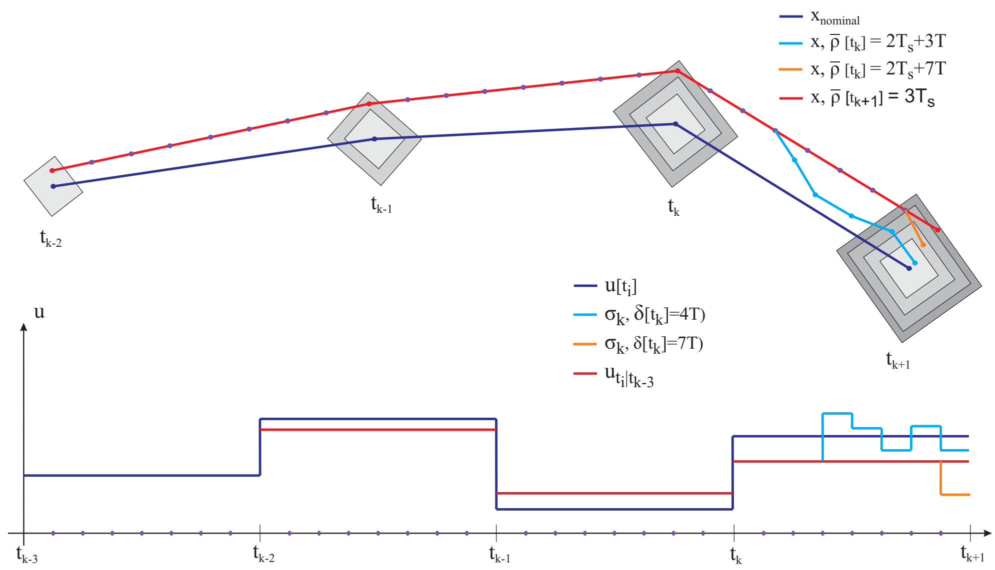


FIGURE 5.11: Compensation of the error caused by time-delay

Proof. Let Hypothesis 5.3 hold. Then, $rank(W) = n$, where W is the following controllability matrix

$$W = \begin{bmatrix} \tilde{B} & \tilde{A}\tilde{B} & \dots & \tilde{A}^{n-1}\tilde{B} \end{bmatrix}.$$

According to controllability index definition (see Appendix), μ is the least integer such that

$$rank(W) = rank(\begin{bmatrix} \tilde{B} & \tilde{A}\tilde{B} & \dots & \tilde{A}^{\mu-1}\tilde{B} \end{bmatrix}) = n.$$

From (5.47) and (5.50) we have that

$$rank\left(\begin{bmatrix} \tilde{A}^{N-\bar{\tau}_{is}[t_k]-1}\tilde{B} & \tilde{A}^{N-\bar{\tau}_{is}[t_k]-2}\tilde{B} & \dots & \tilde{B} \end{bmatrix}\right) = n,$$

i.e., (5.47) and (5.50) are consistent (see Meyer [2000]), if $\bar{\tau}_{is}[t_k] \leq N - \mu$. □

The result which is outlined in the previous theorem is schematically presented on Fig. 5.11 (notice the curve in light blue for $\bar{\tau}_{is}[t_k] \leq N - \mu$ and in orange for $\bar{\tau}_{is}[t_k] > N - \mu$).

Proposition 5.3. *Let μ denote the controllability index of the pair $\left(\tilde{A}, \tilde{B}\right)$ and let*

$$\mathcal{B} = \left\{\tau \mid \tau \in \mathbb{R}_{[0,\tau_{max}]},\ \bar{\tau} = (d-1)T_s + \bar{\tau}_{is}T,\ \bar{\tau}_{is} \in \mathbb{Z}_{[0,N-\mu]}\right\}. \tag{5.52}$$

If $z[t_{k-d[t_k]}] \in \mathcal{Z}$ $(\hat{z}[t_{k-d[t_k]}] \in \hat{\mathcal{Z}})$ *and* $\tau[t_k] \in \mathcal{B}$, $\forall k \in \mathbb{Z}_+$, *then*

$$\exists\sigma_k \in \mathbb{R}^{(N-\bar{\tau}_{is}[t_k])m} \textit{ such that } z[t_{k+1}] \in \mathcal{Z}\ (\hat{z}[t_{k+1}] \in \hat{\mathcal{Z}}), \textit{respectively.}$$

Proof. Let $\tau[t_k] \in \mathcal{B}$ and $z[t_{k-d[t_k]}] \in \mathcal{Z}$ $(\hat{z}[t_{k-d[t_k]}] \in \hat{\mathcal{Z}})$. Then $x[t_{k+1}]$ $(\hat{x}[t_{k+1}])$ can be written as in (5.45) ((5.49)), respectively. According to Theorem 5.1, there exists $\sigma_k \in \mathbb{R}^{(N-\bar{\tau}_{is}[t_k])m}$, obtained from (5.47) ((5.50)), such that (5.48) ((5.51)) holds. □

Remark 5.3. For a chosen N, the Proposition 5.3 determines the set of all delays such that there exists an appropriate compensation vector which governs the tracking error (estimation tracking error) toward a pre-defined bounded region. Such a compensation vector however, does not exist for all delays from the interval $\mathbb{R}_{[0,\tau_{max}]}$. Delays that are not included are determined by $\bar{\tau}_{is} > N - \mu$.

Remark 5.4. If $\bar{\tau}[t_k] = (d[t_k] - 1)T_s + \bar{\tau}_{is}[t_k]T$, $\bar{\tau}_{is}[t_k] > N - \mu$, then $z[t_{k+1}]$ $(\hat{z}[t_{k+1}])$ may not be anymore inside $\mathcal{Z}$ $(\hat{\mathcal{Z}})$, respectively. But, if $\bar{\tau}[t_{k+1}] = \bar{\tau}_{is}[t_{k+1}]T$, $\bar{\tau}_{is}[t_{k+1}] \leq N-\mu$, then the corresponding compensation σ_{k+1} will also take into account the previous tracking (estimation tracking) error that could not be compensated with σ_k. As a consequence, we will have $z[t_{k+2}] \in \mathcal{Z}$ $(\bar{z}[t_{k+2}] \in \bar{\mathcal{Z}})$, respectively, for the price of a more "aggressive" control action.

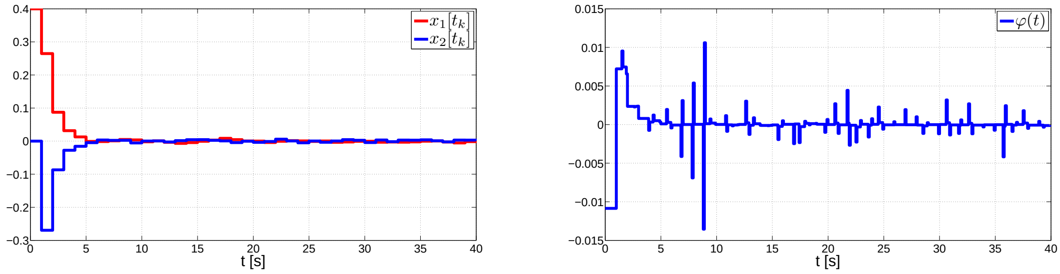
Remark 5.5. Computation of the compensation vector σ is carried out on-line, i.e., linear equations (5.47) and (5.50) need to be solved each time when new measurements are detected by the controller. Nevertheless, computational complexity of such a problem is low since it requires solving a system of linear equations. On the other side, if additional constraints are imposed on the control input (lower and upper bounds for instance), equations (5.47) and (5.50) can be also solved by as a linear programming problem.

Remark 5.6. Since N is a design parameter, it is important to choose an appropriate value for it in order to be able to detect greater range of delays. Theoretically, we can use an arbitrarily high number of inter-sampling periods, but in that case the compensation term for an inter-sampling delay which is arbitrarily close to T_s can have an arbitrarily high magnitude as well. In practical control design however, it is important to find a good compromise between maximal available magnitude of the control action and the corresponding range of inter-sampling delays.

Example 5.3. In order to show how this algorithm performs in a simulation, we consider the same mechanical system of a rolling ball and a beam. For all parameters, we utilize the same values as in Table 5.1 and the same delay variation as shown on Fig. 5.5. Let's keep in mind that the such closed-loop dynamics is unstable for the model-based controller outlined in Section 5.3 (see Fig.5.9).

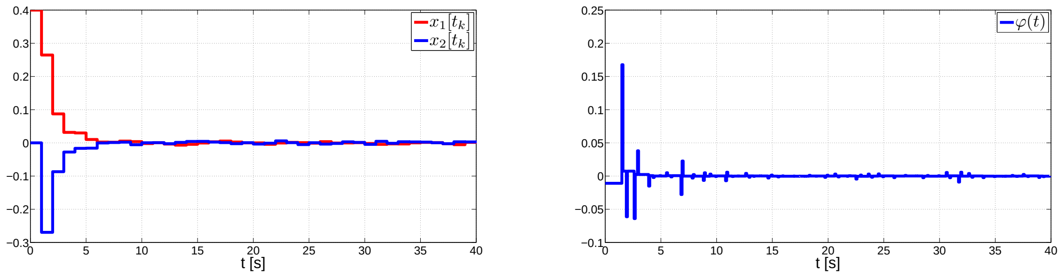
For the same configuration and applied model-based controller with active delay compensation, the response of the closed-loop system is presented on Fig. 5.12.

Let us consider the same control strategy with delay compensation but without the model-based predictor, i.e., let us consider the same control strategy, computed for



(A) Closed-loop response (B) Control signal

FIGURE 5.12: Closed-loop response for prediction-based controller with compensation



(A) Closed-loop response (B) Control signal

FIGURE 5.13: Closed-loop response for nominal controller with compensation

measurements provided with delay of $\tau = 1$ s. The response is shown on Fig. 5.13. Notice that in the second case the control action has much higher magnitude. Such a result can be explained by the fact that in the first case, controller has to compensate only the tracking error caused by the lack of information on additive disturbance which is not taken into account by the predictor. In the second case, the tracking error is larger, since the plant is regulated, based on outdated measurements.

5.5 Set-based delay detection and identification

It is common in networked control systems that data packets are time stamped. If internal clocks in sender and receiver are synchronized, receiver can estimate delay by comparing time stamp with its internal time. Extra network load which is introduced in this way is negligible in comparison to the load caused by transmitting plant's measurements (see Nilsson [1998]). Time stamping also has its negative side: it requires clocks

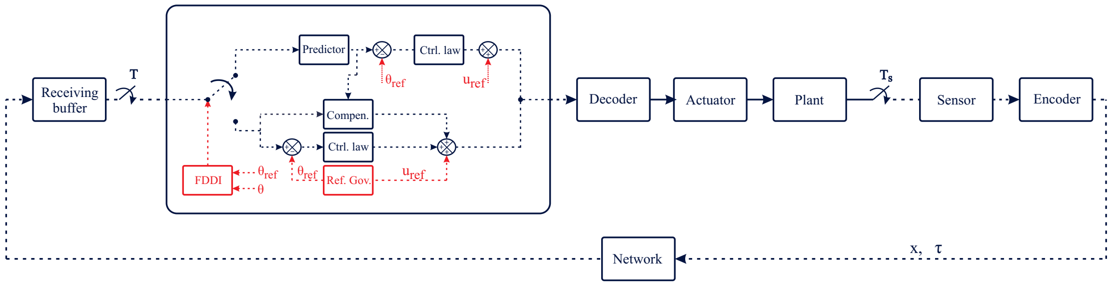


FIGURE 5.14: Model-based controller with delay detection and compensation

synchronization, which is not always easy to achieve. Since clock synchronization represents a wide research area by itself, it will not be considered in this work. Therefore, we assume that all clocks are synchronized (see Hypothesis 2.1).

As it was already stated in the previous section, the model-based controller with state compensation requires information on delay (more precisely, on the time interval between two consecutive inter-samplings determined by $\bar{\tau}_{is}$) in order to generate proper estimation and compensation. This information can be extracted when data packets are time stamped. Such a delay detection strategy is also used in this study when the networked control system is affected by delays such that $\tau > T_s - \mu T$. On the other side, for $\tau \in \mathbb{R}_{[0,T_s-\mu T]}$, we propose a novel method for delay detection which is based on set-membership testing of the corresponding residual signals. The idea for such a delay detection strategy comes from the fault tolerant control approach that was proposed by Stoican et al. [2012] (see Fig. 5.14).

Let us consider a general NCS architecture with a controller that is collocated with the actuator. Due to the fact that the receiving buffer may or may not store the current information (from time t_k), we can define the following residual signals:

Healthy residual: when information which is stored in the buffer is *up-to-date*;

Delayed residual: when information which is stored in the buffer is *outdated*.

Let $\tau \in \mathbb{R}_{[0,T_s-\mu T]}$ and denote by β a value of the buffer. Residual signal is defined as

$$r[t_k + iT] = \beta[t_k + iT] - x_{ref}[t_k] = \theta - x_{ref}[t_k], \tag{5.53}$$

and it is examined whenever the buffer is read.

With regard to information which is stored in the buffer at a given moment, we may have either healthy residual signal

$$r^H[t_k + iT] = \beta[t_k + iT] - x_{ref}[t_k] = \theta[t_k] - x_{ref}[t_k], \tag{5.54}$$

or delayed residual signal

$$r^D[t_k + iT] = \beta[t_k + iT] - x_{ref}[t_k] = \theta[t_{k-1}] - x_{ref}[t_k]. \tag{5.55}$$

In order to have unique detection, healthy and delayed residual signals need to be "different enough" one from another. One can easily notice that such a requirement is not necessarily true, for instance when $\theta[t_{k-1}] = \theta[t_k]$. Therefore, we design a reference governor that will forge a closed-loop response in a way that any two consecutive state vectors are different enough in order to provide delay detection. In the same time, such a modified closed-loop response has to be as close as possible to the initial one.

5.5.1 State feedback residual signals

Let $\theta = x$ and $\beta[t_k + iT] = x[t_k]$, $i \in \mathbb{Z}_{[0,N]}$. By using (5.54), the healthy residual signal is determined as:

$$r^H[t_k + iT] = x[t_k] - x_{ref}[t_k] = z[t_k]. \tag{5.56}$$

On the other side, if $\beta[t_k + iT] = x[t_{k-1}]$, $i \in \mathbb{Z}_{[0,N-\mu]}$, then the corresponding delayed residual signal is given as:

$$r^H[t_k + iT] = z[t_{k-1}] + x_{ref}[t_{k-1}] - x_{ref}[t_k]. \tag{5.57}$$

Let $\tau[t_k] \in \mathbb{R}_{[0,T_s-\mu T]}$ and assume that $z[t_k] \in \mathcal{Z}$. Then, according to Proposition 5.3, $\exists \sigma_k \in \mathbb{R}^{(N-\bar{\tau}_{is}[t_k])m}$ such that $z[t_{k+1}] \in \mathcal{Z}$. In other words, when $z[t_k] \in \mathcal{Z}$, then, by using model-based controller with state compensation, healthy residual signal will always stay within $\mathcal{Z}$ for any delay value from the interval $\mathbb{R}_{[0,T_s-\mu T]}$. Therefore, using (5.56), we can define the following healthy residual set:

$$\mathcal{R}^H = \mathcal{Z}. \tag{5.58}$$

In the similar way, the delayed residual set is determined from (5.57) as:

$$\mathcal{R}^D(x_{ref}) = \{x_{ref}[t_{k-1}] - x_{ref}[t_k]\} \oplus \mathcal{Z}. \tag{5.59}$$

By checking if $r[t_k + iT]$, $i \in \mathbb{Z}_{[0,N]}$, resides in one of these sets, we can affirm that the residual signal is healthy or delayed at $t = t_k + iT$. This provides an unequivocal delay detection and isolation as long as the corresponding residual sets are disjoint:

$$\mathcal{R}^H \cap \mathcal{R}^D(x_{ref}) = \emptyset. \tag{5.60}$$

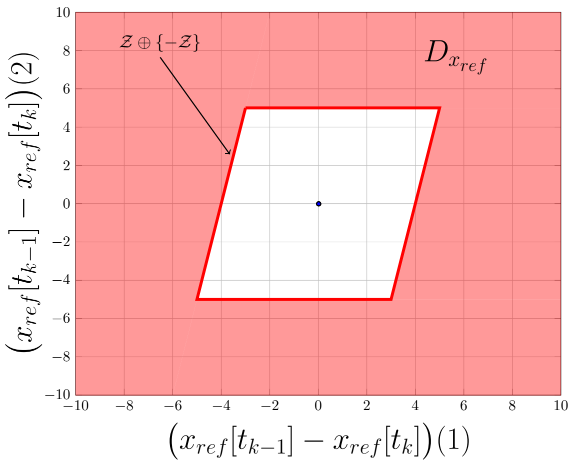


FIGURE 5.15: Reference admissible set

Taking into consideration (5.60), we can define the set of admissible reference states that guarantee the separation between the healthy and delayed residual sets.

$$\mathcal{D}_{x_{ref}} = \{x_{ref}[t_{k-1}], x_{ref}[t_k] \ : \ x_{ref}[t_{k-1}] - x_{ref}[t_k] \in \mathcal{Z} \oplus \{-\mathcal{Z}\}\}\,. \tag{5.61}$$

Since $\mathcal{Z}$ is a C-set, then $\mathcal{D}_{x_{ref}}$ is a non-convex region (see Fig. 5.15).

5.5.2 Estimated state feedback residual signals

Similarly, we define the healthy residual signal with respect to the estimation feedback, i.e., $\theta = \hat{x}$ and $\beta[t_k + iT] = \hat{x}[t_k]$, $i \in \mathbb{Z}_{[0,N]}$.

$$r^H[t_k + iT] = \hat{z}[t_k]. \tag{5.62}$$

On the other side, if $\beta[t_k + iT] = \hat{x}[t_{k-1}]$, $i \in \mathbb{Z}_{[0,N-\mu]}$, then the corresponding delayed residual signal is given as:

$$r^H[t_k + iT] = \hat{z}[t_{k-1}] + x_{ref}[t_{k-1}] - x_{ref}[t_k]. \tag{5.63}$$

Let $\tau[t_k] \in \mathbb{R}_{[0,T_s-\mu T]}$ and assume that $\hat{z}[t_k] \in \hat{\mathcal{Z}}$. Then, according to Proposition 5.3, $\exists \sigma_k \in \mathbb{R}^{(N-\bar{\tau}_{is}[t_k])m}$ such that $\hat{z}[t_{k+1}] \in \hat{\mathcal{Z}}$, i.e., if $\hat{z}[t_k] \in \mathcal{Z}$, then, by using model-based controller with compensation, healthy residual signal will always stay within $\hat{\mathcal{Z}}$ for any delay value from the interval $\mathbb{R}_{[0,T_s-\mu T]}$. Therefore, using (5.62), we can define the following healthy residual set:

$$\mathcal{R}^H = \hat{\mathcal{Z}}. \tag{5.64}$$

In the similar way, the delayed residual set is determined from (5.57) as:

$$\mathcal{R}^D(x_{ref}) = \{x_{ref}[t_{k-1}] - x_{ref}[t_k]\} \oplus \hat{\mathcal{Z}}. \tag{5.65}$$

By checking if $r[t_k + iT]$, $i \in \mathbb{Z}_{[0,N]}$, resides in one of these sets, we can affirm that the residual signal is healthy or delayed at $t = t_k + iT$. This provides an unequivocal delay detection and isolation as long as the corresponding residual sets are disjoint (see (5.60)). Taking into consideration the separation condition, one can define the following set of admissible reference states:

$$\mathcal{D}_{x_{ref}} = \left\{x_{ref}[t_{k-1}], x_{ref}[t_k] \;:\; x_{ref}[t_{k-1}] - x_{ref}[t_k] \in \hat{\mathcal{Z}} \oplus \{-\hat{\mathcal{Z}}\}\right\}. \tag{5.66}$$

5.5.3 Delay identification and reference governor design

If healthy and delayed residual sets are disjoint, by set membership testing, the controller is able to differ up-to-date measurements from the outdated ones. However, this treatment does not provide directly value of $\bar{\tau}_{is}$. In order to extract this information, we assume that delay detection mechanism is equipped with a counter denoted by c. Since all delays are assumed to be smaller than the sampling period, the counter is set to 0 each $t = t_k$, $k \in \mathbb{Z}_+$. Functioning of the counter is described by the following Algorithm 5.1.

Reference governor is designed by using model-based receding horizon optimization framework. The objective of the optimization problem is to design a reference control input u_{ref} which provides a minimal tracking mismatch between an ideal state reference trajectory to be followed, say x_r, and the real reference state (5.4), under certain constraints. The implementation of the reference governor is carried out through as an optimization problem over a finite horizon as follows:

$$u^* = \arg\min_{u_{ref}}\Big(\sum_{q=1}^{s} \|(x_r[t_{k+q}] - x_{ref}[t_{k+q}])\|_Q^2 + \sum_{q=0}^{s-1} \|(u_r[t_{k+q}] - u_{ref}[t_{k+q}])\|_P^2\Big) \tag{5.67}$$

subject to:

$$\begin{aligned} x_{ref}[t_{k+1}] &= Ax_{ref}[t_k] + Bu_{ref}[t_k] \\ x_{ref}[t_k] &\in \mathcal{D}_{x_{ref}}, \quad \forall k \in \mathbb{Z}_+, \end{aligned} \tag{5.68}$$

where

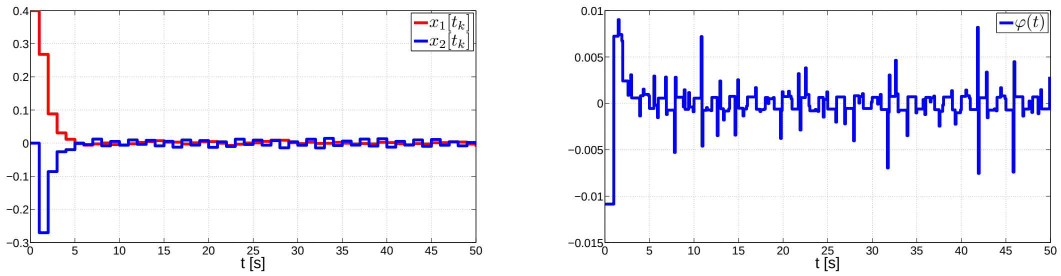
$$x_r[t_{k+1}] = Ax_r[t_k] + Bu_r[t_k], \tag{5.69}$$

represents the ideal reference dynamics to be followed. In (5.67) $s \in \mathbb{Z}^+$ is a prediction horizon while $Q \succ 0$ and $P \succ 0$ are weighting matrices. The reference control action is then set to $u_{ref} = u^*_{[0]}$ which is the first component in the optimal sequence (5.67).

Algorithm 5.1: Single sensor delay identification

Data: $\mathcal{R}^H$, $\mathcal{R}^D$
Input: r
Output: $\bar{\tau}_{is}$

Set $c(0) = 0$;
for $i = 0$ **to** $N - 1$ **do**
  if $r[t_k + iT] \in \mathcal{R}^D$ **then**
    $c(i + 1) = c(i) + 1$;
  else if $r[t_k + iT] \in \mathcal{R}^H$ **then**
    $\bar{\tau}_{is}[t_k] = c(i)$;
    break;
  end
end



(A) Closed-loop response (B) Control signal

FIGURE 5.16: Closed-loop response with reference governor engaged

Remark 5.7. Optimization problem (5.67) is defined over a non convex set of constraints (5.68) which implies that one has to use mixed-integer techniques in order to solve it. Such an optimization method could be numerically complex and inappropriate for an on-line implementation. Luckily enough, if an initial reference input is known in advance (leaving thus enough time for the optimization), this signal can be computed off-line according to (5.67) subject to (5.68).

We end this section by implementing the reference governor in the mechanical system of a rolling ball and a beam that was outlined in the previous examples.

Example 5.4. Consider the system outlined in Example 5.3, where we considered response from initial condition, i.e., for $u_{ref} = 0$. Introducing the reference governor in the loop, the closed-loop response is depicted on Fig. 5.16. One can notice that the input signal provided by the reference governor causes a bigger tracking error with respect to the initial reference input. On the other side, such a tracking error provides the separation between healthy and delayed residual sets, thus unequivocal delay detection.

Remark 5.8. Regarding the closed-loop system response provided in the previous example, we can notice that network induced delays have the biggest impact on stability during the transient response. At the same time, the impact on the tracking error produced by the reference governor is the least. One can use this property to ameliorate the overall tracking performance of the system by temporarily switching off the reference governor while the system is in the steady state and tracking error is confined in a certain region.

5.6 Concluding remarks

It was shown in this chapter that the network induced time-delays in sensor-to-controller communication channel can drastically compromise performance of the closed-loop system. Model-based controller can only partially handle this problem, i.e., only when plant is deterministic i.e. without disturbance. If this is not the case, and the system is not robust with respect to disturbances, then the model-based controller may not succeed in plant stabilization.

To the best of the author's knowledge, there are no proposed results in literature that deal with the problem of communication delays while the process is affected by additive disturbances. That makes the strategy proposed here a first result in this direction. The idea behind this concept is rather simple and natural. First, while the most recent measurements are unavailable, plant is controlled using prediction. At the moment, this is the most relevant available information for generating control action. Then, when appropriate measurements are delivered, controller updates the control input and, in parallel, computes a compensation signal in order to compensate a deviation in state vector caused by using inaccurate feedback information. Estimation obtained from the model-based controller is usually 'more accurate' then delayed measurements. Therefore the error that needs to be compensated is smaller, as consequently the magnitude of the control action.

Another novelty that was brought in by this chapter is detection of "smaller" delays. Namely, by introducing a certain perturbation in the dynamics, we provoke enough difference between healthy and delayed residual signals. This difference, along with the inter-sampling strategy, can indirectly provide information on delay value.

Needless to say that the control and the delay detection strategy are independent one from another. What do they have in common is numerically low cost implementation. Both ideas are also general and can be applied for different problem formulations, for instance in constrained control design or in multi-sensor fault tolerant control design with abrupt faults in sensor-to-controller communication. The second problem is actually what we will consider in the following chapter.

Chapter 6

Multi-sensor NCS with tolerance to abrupt sensor faults

Using redundant sensors in control is inevitable in safety-critical applications. For instance, planes are equipped with several pitot-static systems for measuring airspeed, Mach number and altitude (see, e.g., Eterno et al. [1985]). Nowadays, however, systems with redundant sensors have become a mainstay in many engineering areas due to lower production and operation costs. As examples we can mention cruise control in cars (see Martínez et al. [2006]) or steering and maneuvering control in marine navigation (see Blanke [2006]). The main objective of sensor redundancy in these applications is to provide resilience of the controlled system with respect to an eventual sensor malfunction. Unfortunately, several tragic events (mainly in aeronautics) were initiated by an improper dealing with conflicting measurements (see Flight safety fondation [1999], BEA [2012]). Therefore it is of great importance to design a strategy with fault detection and isolation capabilities which, for instance, could serve as an advisory system (as pointed out in Maciejowski and Jones [2003]).

A shared communication network in a system with redundant sensors can make FTC control more complicated due to network-induced effects such as time-delays and packet dropouts. It is stated in Chapter 2 that network-induced delay is often time-varying. One of the reasons for the delay variability could be the network congestion (see Hespanha et al. [2007]). Taking into account the redundant sensor architecture, one observes that time-delay could be even larger due to an increased use of the network (see Nilsson [1998] for experimental results).

In this study we consider an *active* multi-sensor NCS which guarantees fault tolerance with respect to abrupt sensor faults. Fault and delay detection and isolation (FDDI) is implemented through set membership testings of appropriate residual signals. Namely, realization of the FDDI is achieved through the separation of residual sets which bound residual signals that correspond to healthy, delayed and faulty information. The separation is enforced by a reference governor, which is designed using the receding horizon optimization framework. The initial idea for this fault tolerant control approach can be found in Seron et al. [2008]. In this chapter, however, we also take into account effects induced by the shared communication network between the sensors and the controller. Even though control realization based on a shared network reduces system wiring, provides more flexibility and lower maintenance cost, network-induced delay can significantly affect the functioning of the fault detection and isolation mechanism. Furthermore, beside the abrupt faults, performance of the closed-loop dynamics can be degraded by delayed data transmission (for more details see Chapter 5). Therefore, we combine the model-based controller with active delay compensation (see Chapter 5) and the fault detection and isolation mechanism that is provided in Stoican et al. [2012]. The obtained results show that this approach can indeed provide tolerance for sensor abrupt faults in the presence of network-induced effects.

6.1 Nominal multi-sensor NCS description

Consider the control system depicted in Fig. 6.1 where the plant is monitored by M redundant sensors. Communication between the sensors and the controller is carried out by a shared network while the controller is collocated with the actuator. Sensors could have different data transmission rate due to network-induced delays. Moreover, it is assumed that sensors are also subject to occasional or permanent abrupt faults.

The plant is assumed to be in the continuous-time domain, modeled by a LTI differential equation of the following form:

$$\dot{x}(t) = A_c x(t) + B_c u_j(t) + E_c \omega(t), \tag{6.1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u_j \in \mathbb{R}^m$ is the control signal based on information provided by the j^{th} sensor and $\omega \in \mathbb{R}^p$ is the additive disturbance. The disturbance is bounded by a C-set $\mathcal{W} \subset \mathbb{R}^p$, i.e., $\omega \in \mathcal{W}$. Matrices $A_c \in \mathbb{R}^{n\times n}$, $B_c \in \mathbb{R}^{n\times m}$ and $E_c \in \mathbb{R}^{n\times p}$ are constant.

Due to simpler notations, we examine the case where all sensors are capable of measuring complete state vector. Such an assumption, however, can be relaxed so one can consider the estimated state feedback (for more details see Chapter 5).

State vector of the plant is measured periodically each $T_s \in \mathbb{R}^+$, where $T_s = t_{k+1} - t_k$. All sensors are assumed to be static (or with very fast dynamics relative to the plant dynamics) and to satisfy, under nominal functioning, the following observation equation:

$$y_j[t_k] = x[t_k] + \eta_j[t_k], \quad j \in \mathbb{Z}_{[1,M]}. \tag{6.2}$$

Measurement noise $\eta_j \in \mathbb{R}^n$, is bounded by a C-set $\mathcal{N}_j$, i.e., $\eta_j \in \mathcal{N}_j$, $\forall j \in \mathbb{Z}_{[1,M]}$.

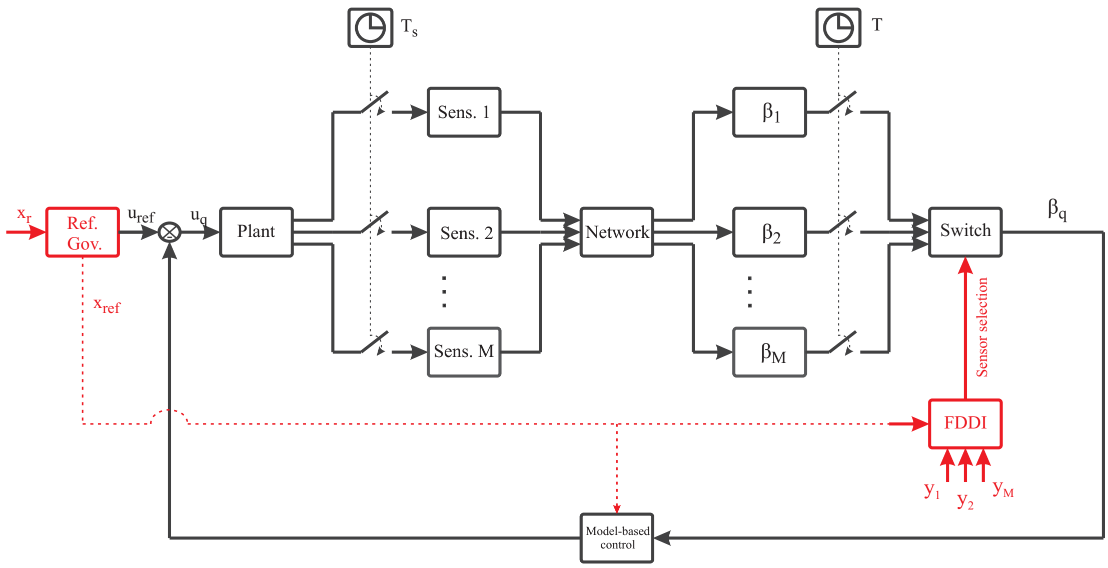


FIGURE 6.1: Multi-sensor control scheme (state feedback architecture)

Measurements acquired by the sensors are transmitted to the controller where they are stored in the receiving buffers. Each feedback channel is followed by a receiving buffer which is assumed to be large enough to store information that corresponds to one sampling period, i.e., data rejection due to buffer overflow is not treated here. All buffers are sampled periodically with the period $T = \frac{T_s}{N}$, where $N \in \mathbb{Z}^+$ is a sufficiently large design parameter. Controller is equipped by a switching mechanism that selects one buffer for computing the control signal each inter-sampling instant. According to available information, the decision which buffer will be employed is made by the FDDI mechanism which discards information coming from the faulty sensors. Moreover, the FDDI mechanism also signalizes when available measurements are outdated so that the controller can properly adapt the control action.

Let $\beta_j[t_k + iT]$, $j \in \mathbb{Z}_{[1,M]}$, denote a value of the j^{th} buffer at $t = t_k + iT$, where $i \in \mathbb{Z}_{[0,N-1]}$. Regarding stored information at each inter-sampling instant, the buffers can be classified into one of the following groups:

- $\mathcal{I}^D[t_k + iT] = \Big\{ j \in \mathbb{Z}_{[1,M]} \ : \ \beta_j[t_k + iT] = y_j[t_{k-1}] \Big\}$;
- $\mathcal{I}^F[t_k + iT] = \Big\{ j \in \mathbb{Z}_{[1,M]} \ : \ \beta_j[t_k + iT] = \eta_j^F, \ \eta_j^F \in \mathcal{N}_j^F \Big\}$;
- $\mathcal{I}^H[t_k + iT] = \Big\{ j \in \mathbb{Z}_{[1,M]} \ : \ \beta_j[t_k + iT] = y_j[t_k] \ \wedge \ j \notin \mathcal{I}^F[t_k + sT], \ \forall s \in \mathbb{Z}_{[0,i)} \Big\}$;
- $\mathcal{I}^R[t_k + iT] = \Big\{ j \in \mathbb{Z}_{[1,M]} \ : \ \beta_j[t_k + iT] = y_j[t_k] \ \wedge \ j \in \mathcal{I}^F[t_k + sT], \ s \in \mathbb{Z}_{[0,i)} \Big\}$,

where $\mathcal{N}_j^F \subset \mathbb{R}^n$ is a C-set. Sets of indices $\mathcal{I}^D$, $\mathcal{I}^F$ and $\mathcal{I}^H$ correspond to buffers with *outdated*, *faulty* and *healthy* measurements. On the other side, set $\mathcal{I}^R$ corresponds to sensors which are under recovery. Notice that if $j \in \mathcal{I}^H[t_k + iT]$ and $j \in \mathcal{I}^R[t_k + iT]$, then $j \in \mathcal{I}^H[t_k + sT]$ and $j \in \mathcal{I}^R[t_k + sT]$, $\forall s \in \mathbb{Z}_{[i,N-1]}$, respectively.

Hypothesis 6.1. Let $j \in \mathcal{I}^H[t_k + sT]$, $s \in \mathbb{Z}_{[0,N-1]}$. If the switching mechanism select β_j at $t = t_k + sT$, then the same buffer is also used at $t = t_k + iT \quad \forall i \in \mathbb{Z}_{[s,N-1]}$.

Regarding Hypothesis 6.1 we say that the closed-loop dynamics is nominal if and only if $\mathcal{I}^H[t_k] \neq \emptyset$. Using zero-order hold sampling with the period T_s the nominal discrete-time representation of (6.1) is written as:

$$x[t_{k+1}] = Ax[t_k] + Bu_j[t_k] + E\omega[t_k], \tag{6.3}$$

In order to select one buffer among several buffers with healthy measurements, one can use additional criterion such as:

$$\beta_j = \arg\min_s \Big(J(\beta_s) \ : \ s \in \mathcal{I}^H \Big), \tag{6.4}$$

where J is a cost function. The same criterion can be used when $\mathcal{I}^H = \emptyset$ and the FDDI mechanism has to select one buffer (among several) with outdated measurements.

The control objective is, for the state of the plant (6.1), to track a reference signal x_{ref} that satisfies the reference dynamics

$$x_{ref}[t_{k+1}] = Ax_{ref}[t_k] + Bu_{ref}[t_k], \tag{6.5}$$

where u_{ref} is the reference control signal.

Let $j \in \mathcal{I}^H[t_k]$. The nominal control law is represented by the following equation:

$$u_j[t_k] = u_{ref}[t_k] - K\left(y_j[t_k] - x_{ref}[t_k]\right). \tag{6.6}$$

The reference control signal can represent a nominal stabilizing control input when the reference dynamics (6.5) is not stable. In (6.6) we utilized the LQR with the gain K (see Section 5.2.1) although in general one can use any other stabilizing controller.

In order to evaluate the control performance, the tracking error $z = x - x_{ref}$ is defined. Taking into consideration (6.6), (6.5) and (6.3), the nominal dynamics admits the following tracking error system:

$$z[t_{k+1}] = (A - BK)z[t_k] - BK\eta_j[t_k] + E\omega[t_k]. \tag{6.7}$$

For a Schur matrix $A - BK$ and bounded process disturbance and measurement noise $\omega \in \mathcal{W}$ and $\eta_j \in \mathcal{N}_j$, respectively, the tracking error dynamics (6.7) admits a RPI set $\mathcal{Z}$

such that:

$$z[t_{k+1}] \in \mathcal{Z}, \quad \forall z[t_k] \in \mathcal{Z}, \quad \forall \eta_j \in \mathcal{N}_j, \quad \forall \omega[t_k] \in \mathcal{W}. \tag{6.8}$$

Notice that if $\mathcal{W}$ and $\mathcal{N}_j$ are C-sets, then $\mathcal{Z}$ is a C-set as well. Construction of RPI sets for linear discrete-time systems with additive disturbance is detailed in Chapter 3.

6.1.1 Fault scenario

Any sensor is prone to fault which can be either temporary (e.g. change in operating conditions) or permanent (e.g. physical damage of the component). A fault can be defined formally as an instantaneous transition between the *healthy mode* of functioning (as given in (6.2)) and a *faulty mode* of functioning[1]:

$$y_j[t_k] = x[t_k] + \eta_j[t_k] \underset{RECOVERY}{\overset{FAULT}{\rightleftharpoons}} y_j[t_k] = \Pi_j x[t_k] + \eta_j^F[t_k]. \tag{6.9}$$

The signature matrix $\Pi_j \in \mathbb{R}^{n \times n}$ represents the loss of effectiveness in the output signal for a given sensor. Moreover, the faulty noise $\eta_j^F[t_k] \in \mathcal{N}_j^F \subset \mathbb{R}^n$ can be used to model nonlinear aberrations, stochastic parameter variations or biases. Arguably, everything that may affect the sensor can be put "under the rug" by using the bounded noise η_j^F (of course, as long as the fault induced phenomena are bounded). While having only partial output failure through a given fault signature matrix covers many cases, for the simplicity of the presentation we consider hereafter only total output outages (i.e., $\Pi_j = \mathbf{0}$):

$$y_j[t_k] = x[t_k] + \eta_j[t_k] \underset{RECOVERY}{\overset{FAULT}{\rightleftharpoons}} y_j[t_k] = 0 \cdot x[t_k] + \eta_j^F[t_k]. \tag{6.10}$$

The fault scenario considered in this thesis is briefly outlined as follows:

- Sensors are prone to abrupt faults as defined as in (6.10).
- Data transmission from sensors to the controller is subject to random and time-varying delay which is less than the sampling period.
- Sets $\mathcal{I}^D[t_k + iT] \neq \emptyset \ \ \forall i \in \mathbb{Z}_{[0,N-\mu)}$ and $\mathcal{I}^H[t_k + (N-\mu)T] \neq \emptyset$, where $\mu \in \mathbb{Z}^+$ is the controllability index of the pair $(\tilde{A}, \tilde{B})$.

Remark 6.1. The abruptness hypothesis can be discarded in favor of faults which describe a gradual output decay. However, none of these elements are conceptually different from the scenario described in (6.10) in the sense that, no new insight in the treatment of the FTC mechanisms can be gained by using the more complex cases. As such, for the brevity of the presentation, we keep with the basic case described by the scenario (6.10).

[1] Depending on the sense of the switch we have a failure or a recovery event.

Remark 6.2. In Seron et al. [2008], Olaru et al. [2010], Stoican et al. [2012], the authors imposed the assumption that there is always available at least one functional sensor. Since an active fault tolerant control strategy is considered (not robust control design with respect to faults occurrence), such an assumption is legitimate. However, in the presence of delay, this assumption is not enough to guarantee that the controller will always be supplied by an *appropriate information*. The term "appropriate", can be understood in the context of a feedback information that can be used safely (healthy mode) or at least it can be corrected by means of the control input (see Chapter 5 and model-based controller with delay compensation). Therefore, the proposed fault scenario guarantees that the controller is always provided by an "appropriate" input.

The outlined fault scenario can be relaxed with respect to allowable delay range. Namely, it is also possible to consider delays that are larger than the sampling period. The only requirement is that those delays would have to be bounded. Since we already considered the general case in Chapter 5, such an extension is not provided here and it can be obtained by following the results outlined in that chapter.

6.2 Control design for a multi-sensor NCS

The main objective in this section is to design a control strategy that provides resilience of the closed-loop system with respect to abrupt sensor faults and/or network-induced delay in the sensor-to-controller loop. The main difference of the multi-sensor architecture when compared with the single-sensor scheme from the previous chapter arises due to the switching between feedback channels with different measurement noises. In other words, the switching may introduce a tracking error which needs to be taken into account when compensation vector is calculated.

As denoted before, let $\tilde{A} \in \mathbb{R}^{n\times n}$ and $\tilde{E} \in \mathbb{R}^{n\times p}$ are state and input matrices with respect to the inter-sampling period T. The following assumption will hold throughout this chapter:

Hypothesis 6.2. The pair $(\tilde{A}, \tilde{B})$ is controllable.

In the following lines we consider the case when all sensor-to-controller communication channels are affected by (different) delays, i.e., $\mathcal{I}^H[t_k] = \emptyset$ (for the nominal case see Section 6.1). Assume that

$$\bar{\tau}^j_{is}[t_k] = \min_i \left\{ \bar{\tau}^i_{is}[t_k] \ : \ i \in \mathbb{Z}_{[1,M]} \right\}. \tag{6.11}$$

Without having the up-to-date measurements available, the switching mechanism selects one of the buffers with outdated information. Let $i_1 \in \mathbb{Z}_{[0,N-1]}$ such that $i_1 < \bar{\tau}^j_{is}[t_k]$. Assume that $\beta_{s_1}[t_k + iT]$, where $s_1 \in \mathcal{I}^D$, is selected during the inter-sampling interval determined by $i \in \mathbb{Z}_{[0,i_1]}$. Since $\beta_{s_1}[t_k + iT] = y_{s_1}[t_{k-1}] \quad \forall i \in \mathbb{Z}_{[0,i_1]}$, the following state prediction can be computed by using the available mathematical model of the plant:

$$\theta^{(s_1)}_{t_k|t_{k-1}} = Ay_{s_1}[t_{k-1}] + B\Big(u_{ref}[t_{k-1}] - K\left(y_{s_1}[t_{k-1}] - x_{ref}[t_{k-1}]\right)\Big). \tag{6.12}$$

By using (6.12) in control (6.6) one obtains:

$$x[t_k+(i_1+1)T] = \tilde{A}^{i_1+1}x[t_k]+\sum_{i=0}^{i_1}\tilde{A}^i\tilde{B}\Big(u_{ref}[t_k]-K\left(\theta^{(s_1)}_{t_k|t_{k-1}} - x_{ref}[t_k]\Big)\right)+\sum_{i=0}^{i_1}\tilde{A}^i\tilde{E}\omega[t_k]. \tag{6.13}$$

Notice that we assumed that ω is constant between two consecutive samplings. This assumption can be relaxed by considering the additive disturbance which is constant within the inter-sampling period (see Section 3.2).

Next, assume the s_1^{th} buffer is updated at $t = t_k + (i_1 + 1)T$, but the transmitted measurements are provided by the faulty sensor, i.e., $\beta_{s_1}[t_k + (i_1 + 1)T] = \eta^F_{s_1}[t_k]$. Consequently, the switching mechanism selects another buffer, say β_{s_2}, where $s_2 \in \mathcal{I}^D[t_k + iT], \quad \forall i \in \mathbb{Z}_{[0,i_2]}$ and $i_2 < \bar{\tau}^j_{is}[t_k]$. The corresponding discrete-time representation at $t = t_k + (i_2 + 1)T$ is:

$$\begin{aligned} x[t_k + (i_2+1)T] = \tilde{A}^{i_2+1}x[t_k] &+ \sum_{i=i_2-i_1}^{i_2}\tilde{A}^i\tilde{B}\Big(u_{ref}[t_k] - K\left(\theta^{(s_1)}_{t_k|t_{k-1}} - x_{ref}[t_k]\Big)\right) \\ &+ \sum_{i=0}^{i_2-i_1-1}\tilde{A}^i\tilde{B}\Big(u_{ref}[t_k] - K\left(\theta^{(s_2)}_{t_k|t_{k-1}} - x_{ref}[t_k]\Big)\right) + \sum_{i=0}^{i_2}\tilde{A}^i\tilde{E}\omega[t_k], \end{aligned} \tag{6.14}$$

where $\theta^{(s_2)}$ is computed in the similar way as (6.12).

Due to different measurement noises, switching among buffers with outdated data introduces a *switching error* which we denote by γ:

$$\gamma^{(s_2\leftarrow s_1)}[t_k] = \theta^{(s_2)}_{t_k|t_{k-1}} - \theta^{(s_1)}_{t_k|t_{k-1}}. \tag{6.15}$$

By using (6.15) in (6.14), one obtains:

$$\begin{aligned} x[t_k + (i_2+1)T] = \tilde{A}^{i_2+1}x[t_k] &+ \sum_{i=0}^{i_2}\tilde{A}^i\tilde{B}\Big(u_{ref}[t_k] - K\left(\theta^{(s_2)}_{t_k|t_{k-1}} - x_{ref}[t_k]\Big)\right) \\ &+ \sum_{i=0}^{i_2}\tilde{A}^i\tilde{E}\omega[t_k] + \sum_{i=i_2-i_1}^{i_2}\tilde{A}^i\tilde{B}K\gamma^{(s_2\leftarrow s_1)}[t_k]. \end{aligned} \tag{6.16}$$

In the similar way, the switching mechanism selects another buffer with the outdated measurements when the previously used buffer is updated by a faulty information. Assume that the s_r^{th} buffer is selected at $t = t_k + iT$, $\forall i \in \mathbb{Z}_{[i_{r-1}+1, \bar{\tau}_{is}^j]}$. Therefore, one can get:

$$
\begin{aligned}
x[t_k + \bar{\tau}_{is}^j[t_k]T] = \tilde{A}^{\bar{\tau}_{is}^j[t_k]} x[t_k] &+ \sum_{i=0}^{\bar{\tau}_{is}^j[t_k]-1} \tilde{A}^i \tilde{B}\Big(u_{ref}[t_k] - K\left(\theta_{t_k|t_{k-1}}^{(s_r)} - x_{ref}[t_k]\Big)\right) \\
&+ \sum_{i=0}^{\bar{\tau}_{is}^j[t_k]-1} \tilde{A}^i \tilde{E}\omega[t_k] + \sum_{i=\bar{\tau}_{is}^j[t_k]-1-i_{r-1}}^{\bar{\tau}_{is}^j[t_k]-1} \tilde{A}^i \tilde{B} K \gamma^{(s_r \leftarrow s_{r-1})}[t_k] + \dots \\
&+ \sum_{i=\bar{\tau}_{is}^j[t_k]-1-i_1}^{\bar{\tau}_{is}^j[t_k]-1} \tilde{A}^i \tilde{B} K \gamma^{(s_2 \leftarrow s_1)}[t_k],
\end{aligned}
\tag{6.17}
$$

where $\gamma^{(s_r \leftarrow s_{r-1})}$, $\gamma^{(s_{r-1} \leftarrow s_{r-2})}$, $\dots$, $\gamma^{(s_2 \leftarrow s_1)}$ are the switching errors computed according to (6.15).

Since $j \in \mathcal{I}^H[t_k + \bar{\tau}_{is}^j T]$, the switching mechanism selects $\beta_j[t_k + \bar{\tau}_{is}^j T] = y_j[t_k]$. According to Hypothesis 6.1, once the buffer with the healthy information is selected, it is employed until the end of the sampling period. The corresponding state vector is given as:

$$
\begin{aligned}
x[t_{k+1}] = Ax[t_k] &+ \sum_{i=N-\bar{\tau}_{is}^j[t_k]}^{N-1} \tilde{A}^i \tilde{B}\Big(u_{ref}[t_k] - K\left(\theta_{t_k|t_{k-1}}^{(s_r)} - x_{ref}[t_k]\Big)\right) + E\omega[t_k] \\
&+ \sum_{i=N-1-i_{r-1}}^{N-1} \tilde{A}^i \tilde{B} K \gamma^{(s_r \leftarrow s_{r-1})}[t_k] + \dots + \sum_{i=N-1-i_1}^{N-1} \tilde{A}^i \tilde{B} K \gamma^{(s_2 \leftarrow s_1)}[t_k] \\
&+ \sum_{i=0}^{N-\bar{\tau}_{is}^j[t_k]-1} \tilde{A}^i \tilde{B}\Big(u_{ref}[t_k] - K(y_j[t_k] - x_{ref}[t_k]) + \sigma[t_k + (N-1-i)T]\Big),
\end{aligned}
\tag{6.18}
$$

where σ is the compensation term which is introduced with control signal computed for the healthy information.

Let $\varepsilon_j \in \mathbb{R}^n$ denote the prediction error with respect to the j^{th} buffer which is defined by the following difference:

$$
\varepsilon_j[t_k] = y_j[t_k] - \theta_{t_k|t_{k-1}}^{(j)}. \tag{6.19}
$$

This information becomes available immediately when the up-to-date information is provided by one of the sensors. By using (6.19) and

$$
\theta_{t_k|t_{k-1}}^{(s_r)} = \theta_{t_k|t_{k-1}}^{(j)} - \gamma^{(j \leftarrow s_r)}[t_k]
$$

in (6.18), one obtains:

$$
\begin{aligned}
x[t_{k+1}] = &\underbrace{Ax[t_k] + Bu_j[t_k] + E\omega[t_k]}_{\text{Nominal dynamics}} + \sum_{i=N-\bar{\tau}_{is}^{j}[t_k]}^{N-1} \tilde{A}^i\tilde{B}K\varepsilon_j[t_k] + \sum_{i=N-\bar{\tau}_{is}^{j}[t_k]}^{N-1} \tilde{A}^i\tilde{B}K\gamma^{(j\leftarrow s_r)}[t_k] \\
&+ \sum_{i=N-1-i_{r-1}}^{N-1} \tilde{A}^i\tilde{B}K\gamma^{(s_r\leftarrow s_{r-1})}[t_k] + \ldots + \sum_{i=N-1-i_1}^{N-1} \tilde{A}^i\tilde{B}K\gamma^{(s_2\leftarrow s_1)}[t_k] \\
&+ \sum_{i=0}^{N-\bar{\tau}_{is}^{j}[t_k]-1} \tilde{A}^i\tilde{B}(\sigma[t_k + (N-1-i)T]),
\end{aligned}
\tag{6.20}
$$

where

$$\sigma_k = \Big[\sigma[t_k + \bar{\tau}_{is}^{j}[t_k]T]^T \quad \ldots \quad \sigma[t_k + (N-1)T]^T\Big] \in \mathbb{R}^{(N-\bar{\tau}_{is}^{j}[t_k])m} \tag{6.21}$$

is the compensation vector.

Regarding the discrete-time system(6.20), one can notice that $\gamma^{(j\leftarrow s_r)}[t_k], \ldots \gamma^{(s_2\leftarrow s1)}[t_k]$ and $\varepsilon_j[t_k]$ become known parameters when up-to-date information is provided to the controller. In order to be able to compute the compensation vector (6.21), it is crucial to know value of $\bar{\tau}_{is}^{j}$. This value can be assessed by the controller either by using time-stamps (see e.g. Nilsson [1998]) or by the FDDI mechanism which is described in the subsequent section.

In order to compensate the tracking error caused by delay in the sensor-to-controller loop, there must exist a compensation vector σ_k such that:

$$
\begin{aligned}
&\sum_{i=0}^{N-\bar{\tau}_{is}^{j}[t_k]-1} \tilde{A}^i\tilde{B}\sigma[t_k + (N-1-i)T] = -\sum_{i=N-\bar{\tau}_{is}^{j}[t_k]}^{N-1} \tilde{A}^i\tilde{B}K\Big(\varepsilon_j[t_k] + \gamma^{(j\leftarrow s_r)}[t_k]\Big) \\
&-\sum_{i=N-1-i_{r-1}}^{N-1} \tilde{A}^i\tilde{B}K\gamma^{(s_r\leftarrow s_{r-1})}[t_k] - \ldots - \sum_{i=N-1-i_1}^{N-1} \tilde{A}^i\tilde{B}K\gamma^{(s_2\leftarrow s_1)}[t_k].
\end{aligned}
\tag{6.22}
$$

As it was outlined in Chapter 5, solution to the linear equation (6.22) may not be feasible. In order to characterize the existence of the compensation vector, we provide the following results.

Theorem 6.1. *Let $\mu \in \mathbb{Z}^+$ denote the controllability index of the pair $(\tilde{A}, \tilde{B})$. Then, the linear equations (6.22) is consistent if $\bar{\tau}_{is}^{j}[t_k] \leq N - \mu$.*

Proof. See the proof of Theorem 5.1. □

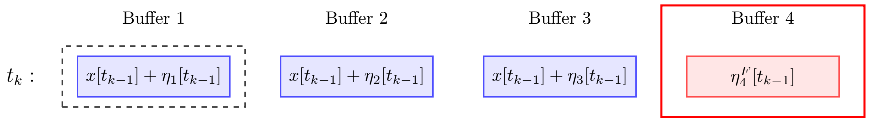


FIGURE 6.2: Contents of receiving buffers at $t = t_k + T$

Remark 6.3. Computation of σ_k is carried out on-line, i.e., the linear equation (6.22) needs to be solved each time when up-to-date measurements are transmitted to the controller. Nevertheless, numerical complexity of solving such an equation is low.

Remark 6.4. If constraints are imposed on control signal, (6.22) can be solved as an optimization problem.

Remark 6.5. Compensation $\sigma[t_k + \bar{\tau}_{is}^j T], \ldots, \sigma[t_k + (N-1)T]$ can also be determined by solving the following optimization problem:

$$\sigma^* = \arg\min_{\sigma} \sum_{i=0}^{N-\bar{\tau}_{is}^j-1} \left(\left\| x[t_k + (\bar{\tau}_{is}^j + i)T] \right\|_Q^2 + \left\| u[t_k] + \sigma[t_k + (\bar{\tau}_{is} + i)T] \right\|_R^2 \right) \\ + x^T[t_{k+1}]Px[t_{k+1}] \tag{6.23}$$

subject to:

$$x[t_{k+1}] \in \{x_{ref}[t_{k+1}]\} \oplus \mathcal{Z} \ominus \sum_{i=1}^{N-\bar{\tau}_{is}-1} \tilde{A}^i \tilde{E} \mathcal{W}. \tag{6.24}$$

In this way it is possible to achieve better performance of the closed-loop system since the terminal constraint (x_{ref} in (6.22)) can be replaced by a set (terminal set constraint in (6.24)) which includes x_{ref}. Operation $\ominus$ denotes the Pontryagin difference (see Chapter 3).

In order to clarify the notations[2] outlined in this section, we consider a simple example step by step. Namely, assume that the process is monitored by four redundant sensors and that $T_s = 4T$. At $t = t_k$, the buffers store measurements as shown on Fig. 6.4. It can be noticed (see Fig. 6.2) that the fourth buffer is discarded immediately since it stores the faulty information from the previous sampling instant. From the rest available buffers with outdated measurements, the switching mechanism selects (according to adopted criteria) the first buffer. The resulting state vector at $t = t_k + T$ is given by the following difference equation:

$$x[t_k + T] = \tilde{A}x[t_k] + \tilde{B}\Big(u_{ref}[t_k] - K(\theta^{(1)}_{t_k|t_{k-1}} - x_{ref}[t_k])\Big) + \tilde{E}\omega[t_k]. \tag{6.25}$$

At the following inter-sampling instant, content of some buffers may be changed depending on the network-induced delays. Let assume that the first buffer receives a

[2]Someone may say cumbersome notations.

faulty sensor information (see Fig. 6.3). Therefore it will be immediately discarded and the switching mechanism will select the other available buffer, say $\beta_2[t_k+T] = y_2[t_{k-1}]$. The resulting state vector at $t = t_k + 2T$ is given by the following difference equation:

$$\begin{aligned} x[t_k+2T] = \tilde{A}^2 x[t_k] + \sum_{i=0}^{1} \tilde{A}^i \tilde{B}\Big(u_{ref}[t_k] - K(\theta^{(2)}_{t_k|t_{k-1}} - x_{ref}[t_k])\Big) \\ + \sum_{i=0}^{1} \tilde{A}^i \tilde{E}\omega[t_k] + \tilde{A}\tilde{B}K\gamma^{(2\leftarrow 1)}[t_k]. \end{aligned} \tag{6.26}$$

From Fig. 6.3 we can also notice that the fourth buffer is updated by the healthy information at $t = t_k + T$. However, this buffer cannot be regarded as healthy since it the recovery confirmation is required (see Subsection 6.1.1).

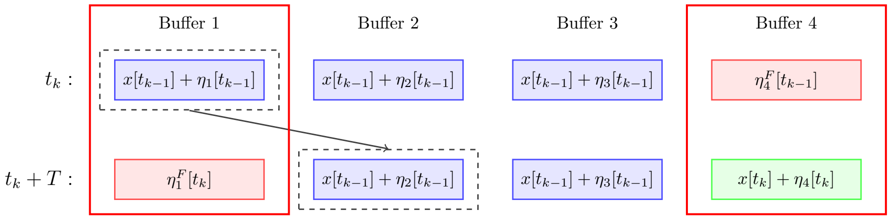


FIGURE 6.3: Contents of receiving buffers at $t = t_k + 2T$

Let for the moment make a digression and assume that the FDDI mechanism anyway selects the fourth buffer at $t = t_k + T$. Then, taking into consideration Hypothesis 6.1, the discrete-time dynamics of the plant is written in the following way:

$$\begin{aligned} x[t_{k+1}] = Ax[t_k] + Bu_4[t_k] + E\omega[t_k] + \tilde{A}^3\tilde{B}K\Big(\varepsilon_4[t_k] + \gamma^{(4\leftarrow 1)}[t_k]\Big) \\ + \sum_{i=0}^{N-2} \tilde{A}^i \tilde{B}\sigma[t_k + (3-i)T], \end{aligned}$$

where ε_4 and $\gamma^{(4\leftarrow 1)}$ are the prediction error and the switching error, respectively. It is clear that both errors are unknown since the measurements of the fourth sensor for the previous sampling instant are unknown. Therefore, these errors may cause a growth of the tracking error on a longer time horizon. For this reason, the fourth sensor cannot be immediately employed in the control loop and the recovery confirmation is required.

Assume that at $t = t_k + 2T$ the third buffer is updated by healthy measurements (see Fig. 6.4). Taking into account Hypothesis 6.1, the state vector of the plant is obtained

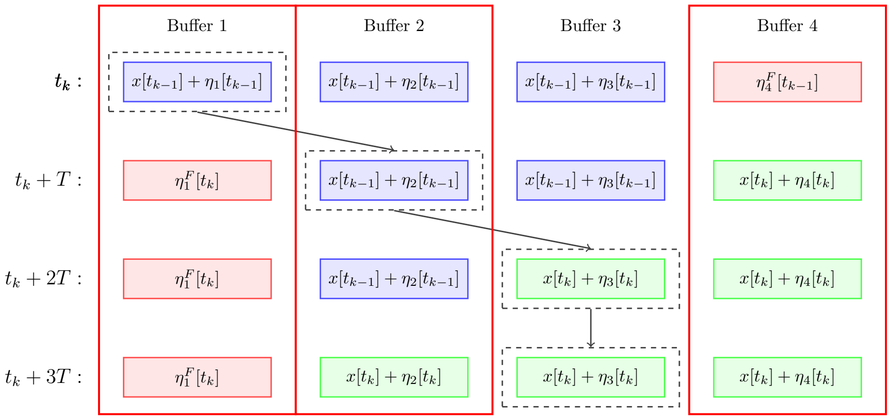


FIGURE 6.4: Contents of receiving buffers at $t = t_k + 3T$

as:

$$x[t_{k+1}] = \underbrace{Ax[t_k] + Bu_3[t_k] + E\omega[t_k]}_{\text{Nominal dynamics}} + \sum_{i=2}^{3} \tilde{A}^i \tilde{B} K \Big(\varepsilon_3[t_k] + \gamma^{(3\leftarrow 2)}[t_k]\Big)$$
$$+ \tilde{A}^3 BK \gamma_{(2\leftarrow 1)}[t_k] + \sum_{i=0}^{1} \tilde{A}^i \tilde{B} \sigma[t_k + (3-i)T].$$

By computing $\varepsilon_3[t_k] = y_3[t_k] - \theta^{(3)}_{t_k|t_{k-1}}$, $\gamma^{(2\leftarrow 1)}[t_k] = \theta^{(2)}_{t_k|t_{k-1}} - \theta^{(1)}_{t_k|t_{k-1}}$ and $\gamma^{(3\leftarrow 2)}[t_k] = \theta^{(3)}_{t_k|t_{k-1}} - \theta^{(2)}_{t_k|t_{k-1}}$, one can determine $\sigma_k = \Big[\sigma[t_k + 2T] \quad \sigma[t_k + 3T]\Big]$.

6.3 Fault detection and isolation in the presence of delay

In order to provide detection of an abrupt sensor fault or an outdated information, one has to define a residual signal and the corresponding *thresholds* that characterize the nominal functioning and data transmission for that sensor. The residual signal should be chosen in such a way that it reflects a fault or delay occurrence in sensor-to-controller communication.

In order to construct residual signals for the considered multi-sensor scheme, we use information from the buffers in combination with the reference dynamics. In other words, the residual signals are constructed in the following way:

$$r_j = \beta_j - x_{ref}, \quad j \in \mathbb{Z}_{[1,M]}. \tag{6.27}$$

In order to perform fault or delay detection, residual signals are compared with their pre-defined thresholds at each inter-sampling instant. Due to the process and the measurement noises, thresholds are characterized by sets. By membership testing one can discern among the examined residual signals which one corresponds to healthy, faulty or delayed measurements.

Recall that the output of a sensor is defined as in (6.2) and it can switch to faulty functioning as in (6.10). However, due to a possible lag induced by the network, controller may receive faulty information with delay. Taking this into account we can define the following residual signals.

Healthy residual: when stored data are *up-to-date* and provided by the functional sensor;

Faulty residual: when stored data are provided by the faulty sensor;

Delayed residual: when stored data are *outdated* and provided by the functional sensor.

While delayed and faulty residuals determine buffers with outdated and faulty measurements, it is important to notice that healthy residual signals may determine buffers with healthy information but also they may correspond to sensors that are under recovery. In order to discern between these cases, one has to keep track of the recovery process for each sensor that used to be faulty.

Depending on a network-induced delay for each sensor-to-controller link, information that is stored in the corresponding buffer can be updated at any moment. However, such an update is noted only at the inter-sampling instants. Let us assume that $j \in \mathcal{I}^H[t_k + iT]$. Then, the corresponding healthy residual signal is determined by:

$$r_j^H[t_k + iT] = \beta_j[t_k + iT] - x_{ref}[t_k] = z[t_k] + \eta_j[t_k]. \tag{6.28}$$

In the same manner, let $j \in \mathcal{I}^F[t_k + iT]$. The corresponding faulty residual signal is obtained as:

$$r_j^F[t_k + iT] = \beta_j[t_k + iT] - x_{ref}[t_k] = \eta_j^F[t_k] - x_{ref}[t_k]. \tag{6.29}$$

The third case concerns functional sensors with data transmission affected by time-varying delay, i.e., $j \in \mathcal{I}^D[t_k + iT]$. The corresponding delayed residual signal is determined by:

$$\begin{aligned} r_j^D[t_k + iT] &= \beta_j[t_k + iT] - x_{ref}[t_k] = x[t_{k-1}] + \eta_j[t_{k-1}] - x_{ref}[t_k] \\ &= z[t_{k-1}] + \eta_j[t_{k-1}] + x_{ref}[t_{k-1}] - x_{ref}[t_k]. \end{aligned} \tag{6.30}$$

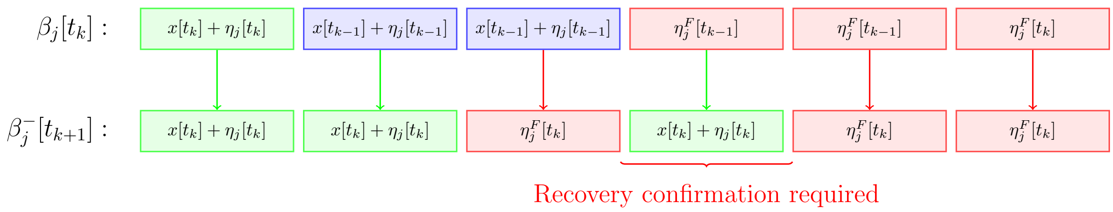


FIGURE 6.5: Possible transitions for a buffer

Remark 6.6. Since η_j^F, $j \in \mathcal{I}^F$, one can notice that there is no need to consider the forth possible combination for the residual signal, i.e., the case when measurements cashed in a buffer are outdated and provided by a faulty sensor. This case corresponds to (6.29) and the corresponding stored measurements are discarded.

Since we consider delay that is smaller than the sampling period, it is simple to notice that each buffer is updated once within two consecutive samplings. Regarding content of a buffer, all possible transitions are depicted on Fig. 6.5.

It is already stated in (6.8) that, for each initial condition from a robust positively invariant set $\mathcal{Z}$, the nominal tracking error dynamics (6.7) stays within the same set. Let us assume that $z[t_k] \in \mathcal{Z}$. By using $\mathcal{Z}$ in (6.28), (6.29) and (6.30), we can construct the corresponding residual sets. Therefore, the bounding set for the healthy residual signals is constructed as:

$$\mathcal{R}_j^H = \mathcal{Z} \oplus \mathcal{N}_j. \tag{6.31}$$

When measurements are provided by a faulty sensor i.e., $\beta_j[t_k + iT] = \eta_j^F[t_k]$ (or $\beta_j[t_k + iT] = \eta_j^F[t_{k-1}]$), the corresponding bounding set is determined as:

$$\mathcal{R}_j^F(x_{ref}) = \{-x_{ref}[t_k]\} \oplus \mathcal{N}_j^F. \tag{6.32}$$

Finally, the bounding set for the delayed residual signals is determined by:

$$\mathcal{R}_j^D(x_{ref}) = \{x_{ref}[t_{k-1}] - x_{ref}[t_k]\} \oplus \mathcal{Z} \oplus \mathcal{N}_j. \tag{6.33}$$

By verifying if $r_j[t_k + iT]$, $i \in \mathbb{Z}_{[0,N-1]}$, resides in one of the sets (6.31)-(6.33), one can affirm that the j^{th} residual signal is healthy, faulty or delayed at $t = t_k + iT$. This provides an unequivocal fault/delay detection and isolation as long as the residual sets are piecewise-disjoint:

$$\mathcal{R}_j^H \cap \mathcal{R}_j^F(x_{ref}) = \emptyset, \quad \mathcal{R}_j^H \cap \mathcal{R}_j^D(x_{ref}) = \emptyset, \quad \mathcal{R}_j^D(x_{ref}) \cap \mathcal{R}_j^F(x_{ref}) = \emptyset. \tag{6.34}$$

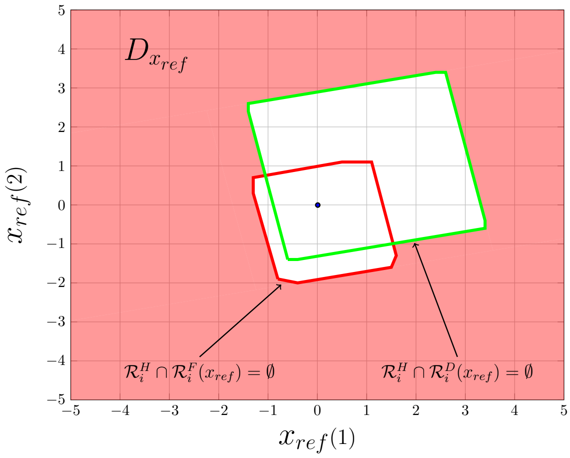


FIGURE 6.6: Admissible domain of reference state

Taking into consideration (6.31), (6.32) and (6.33), the separation condition (6.34) can be directly used in order to define the admissible domain of reference state x_{ref} which allows exact fault detection and isolation:

$$\begin{aligned} \mathcal{D}_{x_{ref}} = \{x_{ref}[t_k], x_{ref}[t_{k-1}] \ : \ & \mathcal{R}_j^H \cap \mathcal{R}_j^F(x_{ref}) = \emptyset, \ \mathcal{R}_j^H \cap \mathcal{R}_j^D(x_{ref}) = \emptyset, \\ & \mathcal{R}_j^D(x_{ref}) \cap \mathcal{R}_j^F(x_{ref}) = \emptyset, \ \forall j \in \mathcal{I}\}. \end{aligned} \tag{6.35}$$

In order to define explicitly the reference admissible region, let us closely consider (6.34). Namely, condition $\mathcal{R}_j^H \cap \mathcal{R}_j^F(x_{ref}) = \emptyset$ can be rewritten as:

$$x_{ref}[t_k] \notin -(\mathcal{Z} \oplus \mathcal{N}_j) \oplus \mathcal{N}_j^F, \quad \forall k \in \mathbb{Z}_+. \tag{6.36}$$

In a similar manner, the separation condition $\mathcal{R}_j^H \cap \mathcal{R}_j^D(x_{ref}) = \emptyset$ can be reformulated in the following form:

$$\{x_{ref}[t_{k-1}] - x_{ref}[t_k]\} \notin -(\mathcal{Z} \oplus \mathcal{N}_j) \oplus \mathcal{Z} \oplus \mathcal{N}_j, \quad \forall k \in \mathbb{Z}_+. \tag{6.37}$$

Finally, $\mathcal{R}_j^D(x_{ref}) \cap \mathcal{R}_j^F(x_{ref}) = \emptyset$ is rewritten as:

$$x_{ref}[t_{k-1}] \notin -(\mathcal{Z} \oplus \mathcal{N}_j) \oplus \mathcal{N}_j^F, \quad \forall k \in \mathbb{Z}_+. \tag{6.38}$$

One can notice that conditions (6.36) and (6.38) are practically equivalent. Therefore, the reference admissible set (6.35) can be simplified by taking into consideration only (6.36) and (6.37) (see Fig. 6.6).

$$\mathcal{D}_{x_{ref}} = \{x_{ref}[t_k], x_{ref}[t_{k-1}] \ : \ \mathcal{R}_j^H \cap \mathcal{R}_j^F(x_{ref}) = \emptyset, \ \mathcal{R}_j^H \cap \mathcal{R}_j^D(x_{ref}) = \emptyset, \ \forall j \in \mathcal{I}\}. \tag{6.39}$$

As long as the invariant set $\mathcal{Z}$ is defined offline, the previous sets can also be described offline and the actual FDDI is a fast online set membership evaluation which differentiates between the healthy, faulty and outdated measurements transmitted through each feedback channel.

The FDDI mechanism discussed here deals with acknowledging the transition from healthy to faulty or healthy to delayed residual signals. To express this in a invariant set-theoretic framework it is necessary to assume that when the testing is done, the auxiliary signals are "where they should be". In other words, the inclusion of the residual signal in one of the sets (6.31), (6.32) or (6.33) makes sense only if the tracking error is in its own bounding sets. Moreover, employing a sensor that is faulty during one sampling interval and which becomes healthy during the subsequent sampling period, introduces a tracking error that cannot be compensated by control action. Therefore, we assume that if a sensor is faulty, it cannot be used immediately after when it admit a healthy residual signal. In other words, a recovery confirmation is required in order to employ that sensor again. In the case of the state feedback recovery of a faulty sensor is rather simple. Namely, since residuals (6.28), (6.29) and (6.30) do not have transient behavior, faulty sensor can be employed again one sampling period after it admits a healthy behavior. The reason for this will be clarified in the subsequent section.

Remark 6.7. Recovery process is far more complex when controller's input is provided by a state observer. In particular, beside verifying the inclusion of the tracking error, one also has to verify the inclusion of the estimation tracking error. Necessary and sufficient conditions for "recovery" in this case, i.e., the mechanism through which the estimation error is validated, were described in a number of papers (note in particular Stoican et al. [2013]). The idea is that through geometrical reasoning or through a combination of over-approximation of the estimation error and subsequent convergence-time estimation, it is possible to guarantee that, after a certain number of iterations under healthy functioning, the estimation error is again inside its set.

6.4 Reference governor design and delay identification

Design of the reference governor is similar as in Chapter 5 and it is carried out by using model-based receding horizon optimization framework. The objective of the optimization problem is to design a reference control input u_{ref} which provides minimal tracking mismatch between an ideal state reference trajectory to be followed x_r, and the real reference state (6.5), under imposed constraints. The implementation of the reference governor is carried out through an offline optimization (for known x_r) over a finite horizon:

$$u^*_{[0,s-1]} = \arg\min_{u_{ref}} \Big(\sum_{i=1}^{s} \|(x_r[t_{k+i}] - x_{ref}[t_{k+i}])\|_Q^2 + \sum_{i=0}^{s-1} \|(u_r[t_{k+i}] - u_{ref}[t_{k+i}])\|_P^2\Big) \quad (6.40)$$

subject to:

$$x_{ref}[t_{k+i+1}] = Ax_{ref}[t_k + i] + Bu_{ref}[t_k + i]$$
$$x_{ref}[t_k + i] \in \mathcal{D}_{x_{ref}}, \quad \forall k \in \mathbb{Z}_+, \quad i \in \mathbb{Z}_{[0,s-1]}, \tag{6.41}$$

where

$$x_r[t_{k+1}] = Ax_r[t_k] + Bu_r[t_k], \tag{6.42}$$

represents the ideal reference dynamics to be followed. In (6.40) $s \in \mathbb{Z}^+$ is a prediction horizon and $Q \succ 0$ and $P \succ 0$ are weighting matrices. The reference admissible set is defined as in (6.39). The reference control action is set to $u_{ref}[t_k] = u^*_{[0]}$ which is the first component in the optimal sequence (6.40). Then, the optimization is reiterated by receding the reference window.

Remark 6.8. Since $\mathcal{D}_{x_{ref}}$ is non-convex set, solving the optimization problem (6.40)-(6.41) requires the mixed-integer programming. More details on this method can be found in Appendix A.

6.4.1 Delay identification

By delay identification we mean evaluation of the minimal delay that is induced by the redundant feedback channels. The main interest of applying the fastest transmitted measurements is the fact that the resulting correction action is distributed over a larger compensation horizon. One can think of such a configuration as competition between redundant sensors where the priority is given to the one with the fastest transmission. Of course, at this stage information provided by faulty or sensors under recovery is discarded by the FDDI mechanism and not taken into account since it is not relevant for computing the control signal.

Identifying delay value (or more precisely, its bounding inter-sampling interval) is important because it allows computation of appropriate compensation vector (see (6.22)). Delay parameter can be evaluated based on knowledge of T_s, N and membership testing of residual signals. Therefore, it is indirectly calculated when a buffer with healthy information is detected by the controller. This principle is outlined by Algorithm 6.1.

6.5 Concluding remarks

By definition, fault-tolerant control is a control design that enables a system to continue its intended operation, possibly at a reduced level, when some part of the system fails. In this chapter, we considered a redundant sensors architecture when sensors are subject to abrupt faults. Moreover, we analyzed control configuration when all sensors are connected with the controller via shared network. In practice, such a configuration

Algorithm 6.1: Multi-sensor delay identification

Data: $\mathcal{R}_j^H,\ \mathcal{R}_j^D,\ j \in \mathcal{I}$
Input: $\beta_j,\ j \in \mathcal{I} \backslash \{\mathcal{I}^F \cup \mathcal{I}^R\}$
Output: $\bar{\tau}_{is}^j[t_k],\ j \in \mathcal{I}^H[t_k + \bar{\tau}_{is}^j[t_k]T]$

Set $c_j(0) = 0$;
for $i = 0$ **to** $N - 1$ **do**
  for $j = 1$ **to** M **do**
    if $r_j[t_k + iT] \in \mathcal{R}_j^D[t_k + iT]$ **then**
      $c_j(i + 1) = c_j(i) + 1$;
    else if $r_j[t_k + iT] \in \mathcal{R}^H[t_k + iT]$ **then**
      $\bar{\tau}_{is}^j[t_k] = c_j(i)$;
      break;
    end
  end
end

usually increases delay in data transmission due to larger number of nodes. This raised the need to provide a control strategy with fault tolerance capabilities not only for abrupt sensors outages, but also with regard to the network-induced delay. The results outlined in this chapter provide a unified fault and delay detection and isolation algorithm for sensors degrading scenarios that are different, i.e., abrupt sensor faults and network-induced delay.

In order to perform fault detection and isolation, we designed a set-based FDI mechanism which is implemented through the set membership testing. We considered information provided by a dysfunctional sensor as worthless for computing the control action. Therefore such sensors were discarded from the control loop. On the other side, information that is provided with delay by a functional sensor was considered still useful. Such information was handled by a model-based controller with delay compensation. It is interesting to notice that, for the conditions outlined in this chapter, even data provided by a broken sensor could be compensated by the model-based controller. In this case however, the control action will exhibit large magnitude in a short correction sub-interval in order to correct the error caused by using inappropriate measurements.

Chapter 7

Conclusions and further directions

THE set-based methods for control design for the discrete-time systems with delays were considered by providing two frameworks in order to address the problems induced by the delays in the control loop.

The first one relies on the positively invariant sets, in particular, D-invariant sets which are robust with respect to the delay parameter. The class of D-invariant sets is more appealing in comparison to the augmented state-space invariant sets when it comes to the computational cost. An important part of the D-invariant notion is its characterization, i.e., definition of constructive necessary and sufficient conditions for the existence. This problem was also addressed in this thesis via strong stability notions for the continuous-time and discrete-time delay difference equations. A computationally efficient numerical test, inspired from the theory of two-dimensional systems, was therefore employed. It was shown that, beside their practical application, D-invariant sets can also bring a new insight on the correlation between the robust asymptotic and the delay-independent stability. This question of the existence is not completely closed yet since the full characterization is still missing. However, we propose the new approach to address this problem which already contributed with computational results that goes beyond the state of the art related to this subject.
Regarding the applications, both classes, D-invariant and invariant sets in the augmented state-space have some disadvantages that can make them ineffective. For D-invariant sets the limitations are related to their delay-independence which makes them applicable only to a restrictive class of dynamics. Therefore an alternative solution based on the factorization is considered and it may represent one possible research direction in order to obtain more flexible set characterization. Moreover, this idea also opens new perspective for a better complexity management of the constraints describing the invariant sets.

The other framework considers a single and multi-sensors architecture for sensors subjected to abrupt faults and delays. Namely, a control configuration when several nodes (sensors) are connected to the controller via shared network increases delays in data transmissions due to the network congestion. This justifies the need to provide a control strategy with fault tolerance capabilities not only for abrupt sensors outages, but also with regard to the network-induced delays. The approach considered in this thesis relies on robust invariant sets which are defined for the nominal dynamics. Delays and abrupt faults are tackled by the set-based controller which is also robust with respect to the exogenous disturbance. Regarding the delay attenuation part, we point out that such a control action is numerically simple to compute and it exploits the inter-sampling strategy. The effectiveness of the controller however will largely depend on imposed constraints. In other words, having the updated feedback information in admissible time interval, one can compute correction action for any prediction error. But in real applications, the possibility to correct certain prediction errors will greatly depend on constraints imposed to the control signal. On the other side, having a controller that is capable of providing control signal of a large magnitude also allows the compensation of larger delays. The most important advantage of such a controller in comparison to the robust controllers is that it may provide stability of a system even for delays for which that system might be unstable with a robust controller. The main disadvantage is that such a control approach can tackle delays that only appear in the sensor-to-controller channel.

Appendices

Appendix A

Non-convex optimization and reference governor design

Let us consider the following optimization problem:

$$u^*_{[0,s-1]} = \arg\min_{u_{ref}} \left\{ \sum_{i=1}^{s} \|x_r[t_{k+i}] - x_{ref}[t_{k+i}]\|^2_Q + \sum_{i=0}^{s-1} \|u_r[t_{k+1}] - u_{ref}[t_{k+1}]\|^2_P \right\} \quad \text{(A.1)}$$

subject to:

$$\begin{aligned} x_r[t_{k+1}] &= Ax_r[t_k] + Bu_r[t_k], \\ x_{ref}[t_{k+j+1}] &= Ax_{ref}[t_k + j] + Bu_{ref}[t_k + j] \\ x_{ref}[t_k + j] &\in \mathcal{D}_{x_{ref}}, \quad \forall k \in \mathbb{Z}_+, \quad j \in \mathbb{Z}_{[1,s]}, \end{aligned} \quad \text{(A.2)}$$

where x_r represents the ideal reference state to be followed. Receding horizon is denoted by $s \in \mathbb{Z}^+$ and the weighting matrices $Q \in \mathbb{R}^{n\times n}$ and $P \in \mathbb{R}^{m\times m}$ are are positive definite.

Cost function (A.1) can be rewritten as:

$$\begin{aligned} u^*_{[0,s-1]} = \arg\min_{u_{ref}} \Big\{ & x^T_{ref}[t_{k+1}|t_{k+s}]\bar{Q}x_{ref}[t_{k+1}|t_{k+s}] - 2x^T_r[t_{k+1}|t_{k+s}]\bar{Q}x_{ref}[t_{k+1}|t_{k+s}] \\ & + u^T_{ref}[t_k|t_{k+s-1}]\bar{P}u_{ref}[t_k|t_{k+s-1}] - 2u^T_r[t_k|t_{k+s-1}]\bar{P}u_{ref}[t_k|t_{k+s-1}] \Big\}, \end{aligned} \quad \text{(A.3)}$$

where $\bar{Q} = BlkDiag(Q, s)$, $\bar{P} = BlkDiag(P, s)$ and

$$\begin{aligned}
x_{ref}^T[t_{k+1}|t_{k+s}] &= \begin{bmatrix} x_{ref}^T[t_{k+1}] & x_{ref}^T[t_{k+2}] & \dots & x_{ref}^T[t_{k+s}] \end{bmatrix}, \\
u_{ref}^T[t_k|t_{k+s-1}] &= \begin{bmatrix} u_{ref}^T[t_k] & u_{ref}^T[t_{k+1}] & \dots & u_{ref}^T[t_{k+s-1}] \end{bmatrix}, \\
x_r^T[t_{k+1}|t_{k+s}] &= \begin{bmatrix} x_r^T[t_{k+1}] & x_r^T[t_{k+2}] & \dots & x_r^T[t_{k+s}] \end{bmatrix}, \\
u_r^T[t_k|t_{k+s-1}] &= \begin{bmatrix} u_r^T[t_k] & u_r^T[t_{k+1}] & \dots & u_r^T[t_{k+s-1}] \end{bmatrix},
\end{aligned}$$

For the receding horizon s, reference state vector is given as:

$$x_{ref}[t_{k+1}|t_{k+s}] = \bar{A}x_{ref}[t_k] + \bar{B}u_{ref}[t_k|t_{k+s-1}], \tag{A.4}$$

where

$$\bar{A} = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^s \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B & 0_{n\times m} & \dots & 0_{n\times m} \\ AB & B & \dots & 0_{n\times m} \\ \vdots & \vdots & \ddots & \vdots \\ A^{s-1}B & A^{s-2}B & \dots & B \end{bmatrix}.$$

Using (A.4) in (A.3) we have:

$$\begin{aligned}
u^*_{[0,s-1]} = &\arg\min_{u_{ref}} \Big\{ u_{ref}^T[t_k|t_{k+s-1}] \left(\bar{B}^T\bar{Q}\bar{B} + \bar{P}\right) u_{ref}[t_k|t_{k+s-1}] \\
&+ (2x_{ref}^T[t_k]\bar{A}^T\bar{Q}\bar{B} - 2x_r^T[t_{k+1}|t_{k+s}]\bar{Q}\bar{B} - 2u_r^T[t_k|t_{k+s-1}]\bar{P})u_{ref}[t_k|t_{k+s-1}] \Big\}.
\end{aligned} \tag{A.5}$$

One can notice that cost (A.5) can be solved by using quadratic programming.

In what follows we reformulate non-convex constraints (A.2) as a set of convex inequality and equality constraints with respect to the reference input u_{ref}.

A.1 $\mathcal{R}^H \cap \mathcal{R}^F(x_{ref}) = \emptyset$

Healthy and faulty residual sets are determined according to the following expression:

$$\mathcal{R}^H = \mathcal{Z}, \quad \mathcal{R}^F = \{-x_{ref}[t_k]\} \oplus \mathcal{N}^F, \; k \in \mathbb{Z}_+. \tag{A.6}$$

Abrupt fault detection and isolation is achieved if $\mathcal{R}^H \cap \mathcal{R}^F = \emptyset$. Condition (A.6) is rewritten as:

$$\{x_{ref}[t_k]\} \notin -\mathcal{Z} \oplus \mathcal{N}^F, \quad \forall k \in \mathbb{Z}_+. \tag{A.7}$$

Let us assume that all sets on the right-hand side in (A.7) are polytopes. Therefore their Minkowski sum is a polytope as well denoted by:

$$\mathcal{P}(F_{HF}, g_{HF}) = -\mathcal{Z} \oplus \mathcal{N}^F,$$

with $F_{HF} \in \mathbb{R}^{p_1 \times n}$, $g_{HF} \in \mathbb{R}^{p_1}$.

Condition (A.7) determines a set of admissible reference states which guarantees $\mathcal{R}^H \cap \mathcal{R}^F = \emptyset$. We denote this admissible set by

$$D_{x_{ref(HF)}} = \{x_{ref}[t_k] \ : \ x_{ref}[t_k] \notin \mathcal{P}(F_{HF}, q_{HF}), \quad k \in \mathbb{Z}_+\}, \tag{A.8}$$

which practically represents complement of the set $\mathcal{P}(F_{HF}, g_{HF})$.

Reference admissible set can be expressed as a set of inequality and equality constraints. For a finite receding horizon $s \in \mathbb{Z}^+$, $D_{x_{ref(HF)}}$ is determined by:

$$\begin{aligned}
-F_{HF}x_{ref}[t_{k+1}] &\leq -(1+\epsilon)g_{HF} + \bar{M}\bar{\alpha}[t_{k+1}], \\
-F_{HF}x_{ref}[t_{k+2}] &\leq -(1+\epsilon)g_{HF} + \bar{M}\bar{\alpha}[t_{k+2}], \\
&\vdots \\
-F_{HF}x_{ref}[t_{k+s}] &\leq -(1+\epsilon)g_{HF} + \bar{M}\bar{\alpha}[t_{k+s}],
\end{aligned} \tag{A.9}$$

and

$$\begin{aligned}
1_{1\times p_1}\bar{\alpha}[t_{k+1}] &= p_1 - 1, \\
1_{1\times p_1}\bar{\alpha}[t_{k+2}] &= p_1 - 1, \\
&\vdots \\
1_{1\times p_1}\bar{\alpha}[t_{k+s}] &= p_1 - 1,
\end{aligned} \tag{A.10}$$

where $\epsilon \in \mathbb{R}^+$ is an arbitrarily small constant and $\bar{M} = MI_{p_1}$ with sufficiently large constant $M \in \mathbb{R}^+$. Vector $\bar{\alpha} = \begin{bmatrix}\alpha(1) & \alpha(2) & \dots & \alpha(p_1)\end{bmatrix}$, where $\alpha(i) \in \{0, 1\}$, $\forall i \in \mathbb{Z}_{[1,p_1]}$.

Using (A.4) in (A.9), we obtain:

$$\begin{bmatrix}-\bar{F}_{HF}\bar{B} & -\tilde{M}\end{bmatrix} \begin{bmatrix} u_{ref}[t_k|t_{k+s-1}] \\ \bar{\alpha}[t_{k+1}|t_{k+s}] \end{bmatrix} \leq -(1+\epsilon)\bar{g}_{HF} + \bar{F}_{HF}\bar{A}x_{ref}[t_k], \tag{A.11}$$

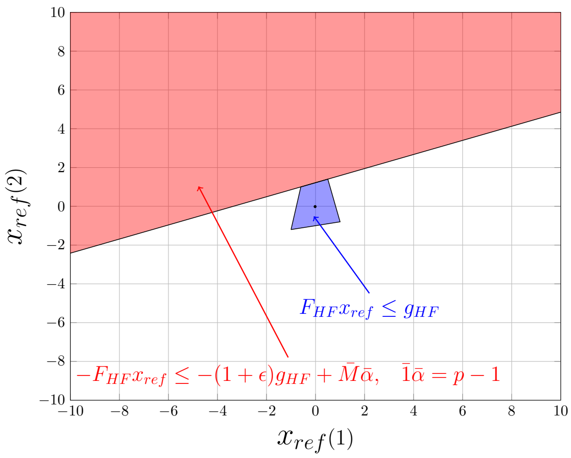


FIGURE A.1: Representing a non-convex region (in blue) by convex sets (in red)

where $\bar{F}_{HF} = BlkDiag(F_{HF}, s)$, $\tilde{M} = BlkDiag(\bar{M}, s)$ and

$$\bar{\alpha}[t_{k+1}|t_{k+s}] = \begin{bmatrix} \bar{\alpha}[t_{k+1}] \\ \bar{\alpha}[t_{k+2}] \\ \vdots \\ \bar{\alpha}[t_{k+s}] \end{bmatrix}, \quad \bar{g}_{HF} = \begin{bmatrix} g_{HF} \\ g_{HF} \\ \vdots \\ g_{HF} \end{bmatrix}.$$

We also redefine the equality constraints (A.10) as:

$$\begin{bmatrix} 0_{s\times ms} & BlkDiag(1_{1\times p_1}, s) \end{bmatrix} \begin{bmatrix} u_{ref}[t_k|t_{k+s-1}] \\ \bar{\alpha}[t_{k+1}|t_{k+s}] \end{bmatrix} = p_1 1_{s\times 1}. \tag{A.12}$$

Transformation of the non-convex bounds from (A.7) into a convex set of constraints is schematically described on Fig. A.1.

A.2 $\mathcal{R}^H \cap \mathcal{R}^D(x_{ref}) = \emptyset$

Healthy and delayed residual sets are defined as:

$$\mathcal{R}^H = \mathcal{Z} \oplus \mathcal{N}, \quad \mathcal{R}^D(x_{ref}) = \{x_{ref}[t_{k-1}] - x_{ref}[t_k]\} \oplus \mathcal{Z} \oplus \mathcal{N}, \quad k \in \mathbb{Z}_+. \tag{A.13}$$

Set of reference states that guarantees separation between healthy and delayed residual sets is obtained as:

$$\{x_{ref}[t_{k-1}] - x_{ref}[t_k]\} \notin -(\mathcal{Z} \oplus \mathcal{N}) \oplus \mathcal{Z} \oplus \mathcal{N}, \quad \forall k \in \mathbb{Z}_+. \tag{A.14}$$

Here we also consider only polytopes. Therefore, the following notation is introduced:

$$\mathcal{P}(F_{HD}, g_{HD}) = -\left(\mathcal{Z} \oplus \mathcal{N}\right) \oplus \mathcal{Z} \oplus \mathcal{N},$$

where $F_{HD} \in \mathbb{R}^{p_2 \times n}$ and $g_{HD} \in \mathbb{R}^{p_2}$.

Using (A.14), the set of admissible reference states that guarantees $\mathcal{R}^H \cap \mathcal{R}^D = \emptyset$ is determined as:

$$D_{x_{ref(HD)}} = \Big\{ x_{ref}[t_{k-1}], x_{ref}[t_k] \;:\; \{x_{ref}[t_{k-1}] - x_{ref}[t_k]\} \notin \mathcal{P}(F_{HD}, q_{HD}),\ \forall k \in \mathbb{Z}_+ \Big\}. \tag{A.15}$$

Let us consider (A.22) on a finite receding horizon $s \in \mathbb{Z}^+$. Such non-convex set can be redefined by using binary variables. As a consequence, one can obtain the following set of inequality and equality constraints:

$$\begin{aligned} -F_{HD}\left\{x_{ref}[t_k] - x_{ref}[t_{k+1}]\right\} &\le -(1+\epsilon) g_{HD} + \bar{M}\bar{\beta}[t_{k+1}], \\ -F_{HD}\left\{x_{ref}[t_{k+1}] - x_{ref}[t_{k+2}]\right\} &\le -(1+\epsilon) g_{HD} + \bar{M}\bar{\beta}[t_{k+2}], \\ &\vdots \\ -F_{HD}\left\{x_{ref}[t_{k+s-1}] - x_{ref}[t_{k+s}]\right\} &\le -(1+\epsilon) g_{HD} + \bar{M}\bar{\beta}[t_{k+s}], \end{aligned} \tag{A.16}$$

and

$$\begin{aligned} 1_{1\times p_2}\bar{\beta}[t_{k+1}] &= p_2 - 1, \\ 1_{1\times p_2}\bar{\beta}[t_{k+2}] &= p_2 - 1, \\ &\vdots \\ 1_{1\times p_2}\bar{\beta}[t_{k+s}] &= p_2 - 1, \end{aligned} \tag{A.17}$$

where $\epsilon \in \mathbb{R}^+$ is an arbitrarily small constant and $\bar{M} = M I_{p_2}$ with sufficiently large constant $M \in \mathbb{R}^+$. Vector $\bar{\beta} = \begin{bmatrix} \beta(1) & \beta(2) & \dots & \beta(p_2) \end{bmatrix}$, where $\beta(q) \in \{0, 1\}$, $\forall q \in \mathbb{Z}_{[1,p_2]}$.

Using (A.4), we can rewrite (A.16) as:

$$\begin{bmatrix} \bar{F}_{HD}\bar{B} & -\tilde{M} \end{bmatrix} \begin{bmatrix} u_{ref}[t_k|t_{k+s-1}] \\ \bar{\beta}[t_{k+1}|t_{k+s}] \end{bmatrix} \le -(1+\epsilon)\bar{g}_{HD} + \left(\tilde{F}_{HD} - \bar{F}_{HD}\bar{A}\right) x_{ref}[t_k], \tag{A.18}$$

where

$$\bar{F}_{HD} = \begin{bmatrix} F_{HD} & 0_{p_2\times n} & \dots & 0_{p_2\times n} & 0_{p_2\times n} \\ -F_{HD} & F_{HD} & \dots & 0_{p_2\times n} & 0_{p_2\times n} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0_{p_2\times n} & 0_{p_2\times n} & \dots & -F_{HD} & F_{HD} \end{bmatrix}, \quad \tilde{F}_{HD} = \begin{bmatrix} F_{HD} \\ 0_{p_2\times n} \\ \vdots \\ 0_{p_2\times n} \end{bmatrix}.$$

Other parameters are defined as in (A.11).

The equality constraints (A.17) are rewritten as:

$$\begin{bmatrix} 0_{s\times ms} & BlkDiag(1_{1\times p_2}, s) \end{bmatrix} \begin{bmatrix} u_{ref}[t_k|t_{k+s-1}] \\ \bar{\beta}[t_{k+1}|t_{k+s}] \end{bmatrix} = p_2 1_{s\times 1}. \tag{A.19}$$

A.3 $\mathcal{R}^D \cap \mathcal{R}^F = \emptyset$

Delayed and faulty residual sets are defined as:

$$\mathcal{R}^D(x_{ref}) = \{x_{ref}[t_{k-1}] - x_{ref}[t_k]\} \oplus \mathcal{Z} \oplus \mathcal{N}, \quad \mathcal{R}^F = \{-x_{ref}[t_k]\} \oplus \mathcal{N}^F, \quad k \in \mathbb{Z}_+. \tag{A.20}$$

This sets are disjoint if and only if the reference state satisfies

$$x_{ref}[t_{k-1}] \notin -(\mathcal{Z} \oplus \mathcal{N}) \oplus \mathcal{N}^F, \quad \forall k \in \mathbb{Z}_+. \tag{A.21}$$

Similarly as it was done before, we assume that all sets are polytopes. Then, we introduce the following notation:

$$\mathcal{P}(F_{DF}, g_{DF}) = -(\mathcal{Z} \oplus \mathcal{N}) \oplus \mathcal{N}^F,$$

where $F_{DF} \in \mathbb{R}^{p_3\times n}$ and $g_{DF} \in \mathbb{R}^{p_3}$.

Using (A.21), we can define a set of admissible reference states in order to guarantee $\mathcal{R}^D \cap \mathcal{R}^F = \emptyset$ as:

$$D_{x_{ref(DF)}} = \Big\{x_{ref}[t_{k-1}] \ : \ x_{ref}[t_{k-1}] \notin \mathcal{P}(F_{DF}, q_{DF}), \ \forall k \in \mathbb{Z}_+\Big\}. \tag{A.22}$$

For a finite receding horizon $s \in \mathbb{Z}^+$, reference admissible set $D_{x_{ref(DF)}}$ is determined by the following set of inequality

$$\begin{aligned} -F_{DF}x_{ref}[t_k] &\leq -(1+\epsilon)g_{DF} + \bar{M}\bar{\gamma}[t_{k+1}], \\ -F_{DF}x_{ref}[t_{k+1}] &\leq -(1+\epsilon)g_{DF} + \bar{M}\bar{\gamma}[t_{k+2}], \\ &\vdots \\ -F_{DF}x_{ref}[t_{k+s-1}] &\leq -(1+\epsilon)g_{DF} + \bar{M}\bar{\gamma}[t_{k+s}], \end{aligned} \tag{A.23}$$

and equality constraints

$$\begin{aligned} 1_{1\times p_3}\bar{\gamma}[t_{k+1}] &= p_3 - 1, \\ 1_{1\times p_3}\bar{\gamma}[t_{k+2}] &= p_3 - 1, \\ &\vdots \\ 1_{1\times p_3}\bar{\gamma}[t_{k+s}] &= p_3 - 1, \end{aligned} \tag{A.24}$$

where $\epsilon \in \mathbb{R}^+$ is an arbitrarily small constant and $\bar{M} = MI_{p_3}$ with sufficiently large constant $M \in \mathbb{R}^+$. Vector $\bar{\gamma} = \begin{bmatrix}\gamma(1) & \gamma(2) & \dots & \gamma(p_3)\end{bmatrix}$, where $\gamma(q) \in \{0,1\}$, $\forall q \in \mathbb{Z}_{[1,p_3]}$).

Using (A.4), we can rewrite (A.23) as:

$$\begin{bmatrix}-\bar{F}_{DF}\bar{B} & -\tilde{M}\end{bmatrix}\begin{bmatrix}u_{ref}[t_k|t_{k+s-1}] \\ \bar{\gamma}[t_{k+1}|t_{k+s}]\end{bmatrix} \leq -(1+\epsilon)\bar{g}_{DF} + \left(\tilde{F}_{DF} - \bar{F}_{DF}\bar{A}\right)x_{ref}[t_k], \tag{A.25}$$

where

$$\bar{F}_{DF} = \begin{bmatrix} 0_{p_3\times n} & \dots & 0_{p_3\times n} & 0_{p_3\times n} \\ F_{DF} & \dots & 0_{p_3\times n} & 0_{p_3\times n} \\ \vdots & \ddots & \vdots & \vdots \\ 0_{p_3\times n} & \dots & F_{DF} & 0_{p_3\times n} \end{bmatrix}, \quad \tilde{F}_{DF} = \begin{bmatrix} F_{DF} \\ 0_{p_3\times n} \\ \vdots \\ 0_{p_3\times n} \end{bmatrix}.$$

Other parameters are defined as in (A.11).

We also redefine the equality constraints (A.17) as:

$$\begin{bmatrix}0_{s\times ms} & BlkDiag(1_{1\times p_3}, s)\end{bmatrix}\begin{bmatrix}u_{ref}[t_k|t_{k+s-1}] \\ \bar{\gamma}[t_{k+1}|t_{k+s}]\end{bmatrix} = p_3 1_{s\times 1}. \tag{A.26}$$

A.4 Reference governor design

Let us consider again the cost function (A.1) and the non-convex constraints (A.2). Taking into account constraints reformulation, outlined in the previous discussion, we can rewrite the optimization problem as a mixed integer quadratic programming:

$$u^*_{[0,s-1]} = \arg\min_{u_{ref}} \{f^T\xi + \frac{1}{2}\xi^T H\xi\}. \tag{A.27}$$

subject to:

$$\begin{aligned} x_r[t_{k+1}] &= Ax_r[t_k] + Bu_r[t_k], \\ x_{ref}[t_{k+j+1}] &= Ax_{ref}[t_k + j] + Bu_{ref}[t_k + j], \\ x_{ref}[t_k + j] &\in \mathcal{D}_{x_{ref}}, \quad \forall k \in \mathbb{Z}_+, \quad j \in \mathbb{Z}_{[1,s]}, \end{aligned} \tag{A.28}$$

where

$$
\begin{aligned}
\xi^T &= \begin{bmatrix} u_{ref}^T[t_k|t_{k+s-1}] & \bar{\alpha}[t_{k+1}|t_{k+s}] & \bar{\beta}[t_{k+1}|t_{k+s}] & \bar{\gamma}[t_{k+1}|t_{k+s}] \end{bmatrix}, \\
H &= \begin{bmatrix} \bar{B}^T\bar{Q}\bar{B} + \bar{P} & 0_{ms\times ps} \\ 0_{ps\times ms} & 0_{ps\times ps} \end{bmatrix}, \\
f^T &= \begin{bmatrix} (2x_{ref}^T[t_k]\bar{A}^T\bar{Q}\bar{B} - 2x_r^T[t_{k+1}|t_{k+s}]\bar{Q}\bar{B} - 2u_r^T[t_k|t_{k+s-1}]\bar{P}) & 0_{1\times ps} \end{bmatrix}, \quad p = p_1 + p_2 + p_3.
\end{aligned}
$$

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Résumé : On considère la synthèse de la commande basée sur un asservissement affecté par des retards. L'approche utilisée repose sur des méthodes ensemblistes. Une partie de cette thèse est consacrée à une conception de commande active pour la compensation des retards qui apparaissent dans des canaux de communication entre le capteur et correcteur. Ce problème est considéré dans une perspective générale du cadre de commande tolérante aux défauts où des retards variés sont vus comme un mode particulier de dégradation du capteur. Le cas avec transmission de mesure retardée pour des systèmes avec des capteurs redondants est également examiné. Par conséquent, un cadre unifié est proposé afin de régler le problème de commande basé sur la transmission des mesures avec retard qui peuvent également être fournies par des capteurs qui sont affectés par des défauts soudains.

Dans la deuxième partie le concept d'invariance positive pour des sysèmes linéaires à retard à temps discret est exposé. En ce qui concerne l'invariance pour cette classe des systèmes dynamiques, il existe deux idées principales. La première approche repose sur la réécriture d'un tel système dans l'espace d'état augmenté et de le considérer comme un système linéaire. D'autre part, la seconde approche considère l'invariance dans l'espace d'état initial. Cependant, la caractérisation d'un tel ensemble invariant est encore une question ouverte, même pour le cas linéaire. Par conséquent, l'objectif de cette thèse est d'introduire une notion générale d'invariance positive pour des sysèmes linéaires à retard à temps discret. Également, certains nouveaux éclairages sur l'existence et la construction pour les ensembles invariants positifs robustes sont détaillés. En outre, les nouveaux concepts d'invariance alternatives sont décrits.

Abstract: We considered the process regulation which is based on feedback affected by varying delays. Proposed approach relies on set-based control methods. One part of the thesis examines active control design for compensation of delays in sensor-to-controller communication channel. This problem is regarded in a general perspective of the fault tolerant control where delays are considered as a particular degradation mode of the sensor. Obtained results are also adapted to the systems with redundant sensing elements that are prone to abrupt faults. In this sense, an unified framework is proposed in order to address the control design with outdated measurements provided by unreliable sensors.

Positive invariance for linear discrete-time systems with delays is outlined in the second part of the thesis. Concerning this class of dynamics, there are two main approaches which define positive invariance. The first one relies on rewriting a delay-difference equation in the augmented state-space and applying standard analysis and control design tools for the linear systems. The second approach considers invariance in the initial state-space. However, the initial state-space characterization is still an open problem even for the linear case and it represents our main subject of interest. As a contribution, we provide new insights on the existence of the positively invariant sets in the initial state-space. Moreover, a construction algorithm for the minimal robust D-invariant set is outlined. Additionally, alternative invariance concepts are discussed.